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The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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ON A FAMILY OF GENERALIZED CONCHOIDS*

R. M. WINGER, University of Washington

Curves with the polar equation $\rho = a \cos p\theta/q + k$ were studied by Moritz† under the name of cyclic-harmonic, using mainly polar coordinates. Following Moritz, Stratton‡ discussed the "polar tangent curves" $\rho = a \tan p\theta/q + k$. The author§ wrote a paper on each of these curves, employing a compact parametric representation. In this paper, which is the third of a trilogy, we consider the remaining case

(1) $\rho = a \sec p\theta/q + k$, a, k real, p, q relatively prime integers.

For our purpose it is convenient to introduce circular or absolute coordinates

$$x \equiv X + iY, \qquad \bar{x} \equiv X - iY, \qquad i^2 = -1,$$

X, Y rectangular, which in turn can be expressed in polar coordinates in the usual way. Then we have

(2) $$x = X + iY = \rho(\cos\theta + i \sin\theta) = (a \sec p\theta/q + k)(\cos\theta + i \sin\theta).$$

Now replace θ by $q\phi$ and let $t \equiv \cos\phi + i \sin\phi$ be the parameter. By De Moivre's theorem

$$\cos p\phi + i \sin p\phi = t^p, \qquad \cos p\phi - i \sin p\phi = t^{-p},$$

whence $2 \cos p\phi = t^p + t^{-p}$. Thus after reduction we get for x and similarly for $\bar{x}$

(3) $$x = \frac{t^q(kt^{2p} + 2at^p + k)}{t^{2p} + 1}, \qquad \bar{x} = \frac{kt^{2p} + 2at^p + k}{t^q(t^{2p} + 1)},$$

which are parametric equations of the curve in circular coordinates. In this representation t is a complex number of absolute value 1, and real points (*i.e.*, those which can be plotted) have conjugate complex coordinates.#

Equations (3) may be made homogeneous in the parameter or in the coordinates or both by the substitutions $t = t_1/t_2$, and $x = x_1/x_3$, $\bar{x} = x_2/x_3$. We get thus

(4) $$\begin{aligned} x_1 &= t_1^{2q}(kt_1^{2p} + 2at_1^p t_2^p + kt_2^{2p}), \\ x_2 &= t_2^{2q}(kt_1^{2p} + 2at_1^p t_2^p + kt_2^{2p}), \\ x_3 &= t_1^q t_2^q (t_1^{2p} + t_2^{2p}). \end{aligned}$$

The triangle of reference then consists of the circular rays, x_1, x_2 and the line

* Read under the title "A generalization of Cotes' spirals," at the sixty-first summer meeting of the American Mathematical Society, Seattle, 1956.

† Univ. Washington Publ. Math., vol. 1, 1923, pp. 1–58.

‡ This MONTHLY, vol. 43, 1936, p. 398.

§ Univ. Washington Publ. Math., vol. 3, No. 1, 1948, and No. 2, 1952.

R. M. Winger, An Introduction to Projective Geometry, Boston, 1923, p. 326.

at infinity, $x_3=0$, with vertices O, the origin, and I, J, the circular points.

The line equations are, removing the factor $2(p+q)(t_1t_2)^{q-1}$,

$$
\begin{aligned}
u_1 &= -kqt^{4p} - 2a(p+q)t^{3p} - 2kqt^{2p} + 2a(p-q)t^p - kq,\\
u_2 &= t^{2q}[-kqt^{4p} + 2a(p-q)t^{3p} - 2kqt^{2p} - 2a(p+q)t^p - kq],\\
u_3 &= 2qt^q(kt^{2p} + 2at^p + k)^2,
\end{aligned} \tag{5}
$$

where we have returned to the nonhomogeneous parameter.

The simple conchoid, $p=q=1$ is a rational circular quartic, having a line of symmetry, a node at the origin, a tacnode at infinity whose tangent is the only real asymptote. We shall find analogues for all these properties for the more general conchoids.

The existence of polynomial parametric equations proves that the general conchoid is rational. The form of (4) and (5) shows that the order in general is $2p+2q$ and the class $4p+2q$. We see also that there is a multiple point O at the origin whose parameters are given by the common factor in x_1 and x_2. Further there is a q-fold point with coincident parameters and $2q$-fold point contact tangent at each of the circular points, absorbing $q-1$ cusps, since $(t_1t_2)^{q-1}$ factors out of the line equations. The other intersections with the line at infinity are double points since $t=\pm i^{1/p}$, where $i^{1/p}$ is any one of the pth roots of i, name the same point.

We ask now whether these are tacnodes. The condition is that the two tangents of each node coincide. Setting $t=i^{1/p}$ in (5), we find as the coordinates of the tangent line of one branch, after removing the factor $4ai$, $(p, -pt^{2q}, 2iaqt^q)$. Next the coordinates of the other branch $t=-i^{1/p}$ are, removing the factor $(-1)^p4ai$, $(p, -p^{2q}, (-1)^p2iaqt^q)$, which differ from the former only in the third coordinate. We now observe that the two sets of coordinates will be the same if and only if p and q are both odd or both even. But they cannot both be even under the original hypotheses, hence we conclude

*The p nodes on the line at infinity whose parameters are $t^{2p}+1=0$ are tacnodes if and only if p and q are both odd.**

Further we note that these nodes are cut out by the p lines $x_1^p+x_2^p=0$ when q is odd and by $x_1^p-x_2^p=0$ when q is even. For, replacing x_1, x_2 by their values from (4) we obtain respectively

$$(kt^{2p} + 2at^p + k)^p(t^{2pq} + 1) \quad \text{and} \quad (kt^{2p} + 2at^p + k)^p(t^{2pq} - 1),$$

the first of which is divisible by $t^{2p}+1$ when q is odd, while the second contains the factor when q is even. Obviously, only one of the nodes can lie on each of the lines.

The flex form, after removing the factor $4q(p+q)(2p+2q-1)^2(t_1t_2)^{2q-2}$ is, returning to the nonhomogeneous form,

* If they were cusps which also have coincident tangents, the parameters would be factors of the line equation.

$$
\begin{aligned}
(6)\qquad & -k^2q^2(t^{6p}+1) - 2ak(p^2+2q^2)(t^{5p}+t^p) \\
& \quad + [4a^2(p^2-q^2) - 3k^2q^2](t^{4p}+t^{2p}) + 4ak(3p^2-2q^2)t^{3p}.
\end{aligned}
$$

The discarded factor indicates that $q-1$ flexes are absorbed at each of the circular points, in addition to the $2q-2$ needed for the cusps at each point. Other cases of coincident flexes will be noted later.

Symmetry. The curve in the general case is invariant under dihedral groups, binary and ternary, of order $2p$. Generators are: of the binary group,

$$
(7)\qquad t' = \epsilon t \quad \text{and} \quad t' = 1/t, \qquad \epsilon = e^{2\pi i/p};
$$

and of the ternary group,

$$
(8)\qquad x' = \epsilon x, \quad \bar{x}' = \epsilon^{-1}\bar{x}, \quad \text{and} \quad x' = \bar{x}, \quad \bar{x}' = x.
$$

The binary group contains p involutions, $t' = \epsilon^k/t$, $k = 1, \cdots, p$ and, when p is even, an extra one $t' = -t$. Likewise the ternary group contains p reflexions

$$
(9)\qquad x' = \epsilon^k\bar{x}, \quad \bar{x}' = \epsilon^{-k}x
$$

and, when p is even, an additional one

$$
(10)\qquad x' = -x, \quad \bar{x}' = -\bar{x},
$$

whose center is the origin and whose axis is the line at infinity. The axes of (9) together are $x^p - \bar{x}^p = 0$, thus forming p equispaced lines on the origin, and the centers lie on the line at infinity. When p is odd the centers lie on the perpendicular set $x^p + \bar{x}^p = 0$. When p is even the centers lie on the axes.

We consider now two subcases of the general case.

p odd. The axes of reflexion are now conjugate as are also the centers. The axis $x - \bar{x} = 0$ cuts out the points $(kt^{2p} + 2at^p + k)(t^{2q} - 1) = 0$, which include ± 1, the fixed points of the involution $t' = 1/t$. These are contacts of tangents from the corresponding center. The other intersections aside from the multiple point, given by $(t^{2q}-1)/(t^2-1)$, form $q-1$ double points. Similarly for the other axes of symmetry, accounting thus for $p(q-1)$ double points. When q is also odd, as shown above, the nodes on the line at infinity are tacnodes and are cut out by the lines $x^p + \bar{x}^p = 0$, in other words they are the centers of reflexion. But when q is even these nodes are ordinary and lie on the axes and are not centers of reflexion. In either case the nodal tangents are asymptotes, real because parallel to the lines $x^p + \bar{x}^p = 0$ or $x^p - \bar{x}^p = 0$, which are all real in this representation. Or

The curve has p or $2p$ real asymptotes according as q is odd or even, one perpendicular to each axis of symmetry when q is odd, but parallel in pairs when q is even.

Each tacnodal tangent is a sort of double asymptote, two branches of the curve approaching it in either direction.*

* *Cf.* the simple conchoid, $p = q = 1$.

p even, q necessarily odd. The curve now admits the extra reflexion (10) and the origin is a center of symmetry. While the parameters of the multiple point O form a general set under the binary group, they are paired in the involution $t' = -t$ so that the tangents at t and $-t$ coincide, hence

The multiple point at the origin contains p tacnodes, absorbing p additional nodes, the p tangents forming a special conjugate set of lines.

There are $2p$ real asymptotes, parallel in pairs, viz., the pairs of tangents of the p nodes on the line at infinity.

Special cases

$a = k$. Equations (3) reduce to

$$x = \frac{kt^q(t^p + 1)^2}{t^{2p} + 1}, \qquad \bar{x} = \frac{k(t^p + 1)^2}{t^q(t^{2p} + 1)} \tag{11}$$

The multiple point O now includes p cusps, parameters $t^p + 1 = 0$.

For x, $\bar{x}$ both contain the factor $(t^p+1)^2$ while the class is reduced by p, t^p+1 factoring out of the line equations.

From symmetry properties we infer that the axes must be cusp tangents. Now the axes are conjugate when p is odd but divide into two sets of $p/2$ lines when p is even. We verify readily:

If p is odd the cusp tangents are the axes of symmetry. But if p is even the cusp tangents coincide in pairs forming double cusp tangents which comprise half the axes, viz., $x^{p/2} + \bar{x}^{p/2} = 0$.

The asymptotic properties remain the same as in the earlier cases when $a \neq k$.

There are cusps at $t^p - 1$ when $a = -k$, the curve however being projectively equivalent to the case $a = k$. Since each cusp absorbs two flexes, the cusp parameters must factor doubly out of the flex form. We verify that $(t^p \pm 1)^2$ is a factor of the flex form when $a = \pm k$, in other words $k^2 - a^2$ is a factor of the discriminant of the flex form. We can get another square factor of the flex form when undulations occur. We saw above that $t = 1$ lies on the axis $x - \bar{x} = 0$ and is a contact of tangent from the corresponding center. The tangent at this point is $x + \bar{x} = 2(a+k)$ which cuts the curve doubly at $t = 1$. If we ask that the tangent cut out the point $t = 1$ triply, then from symmetry it must cut it out quadruply. The condition for triple intersection is $k/a = (p^2 - q^2)/q^2$. The property must be shared by each point conjugate to 1. We conclude that

There are undulations at $t^p - 1 = 0$ *when* $k/a = (p^2 - q^2)/q^2$.

Likewise we find that there are undulations at $t^p + 1 = 0$ when $k/a = -(p^2 - q^2)/q^2$ and that $(t^p \mp 1)^2$ is a factor of the flex form when $k/a = \pm(p^2 - q^2)/q^2$, where upper signs are to be taken together. Thus another factor of the discriminant of the flex form is $q^4k^2 - (p^2 - q^2)^2a^2$.

$q=1$. The circular rays x, $\bar{x}$ now have simple contact at the circular points. We can readily account for all the double points of which the quota is $2p^2+p$. For the $2p$-fold point O is normally equivalent to $2p^2-p$ simple nodes. If p is odd the p double points at infinity are tacnodes, each equivalent to two simple nodes. These with the nodes in O make up the full complement. If p is even the nodes at infinity are ordinary but there are then p tacnodes in O so that again all nodes are accounted for. The equation of the p tacnodal tangents in O when p is even is $kx^p+2a(x\bar{x})^{p/2}+k\bar{x}^p=0$.

$k=0$. These curves are degenerate, the order reducing to $2p$ or $2q$ according as $p>q$ or $p<q$. If however p and q are both odd, the parameters occur only in even powers. Then, replacing t_1^2, t_2^2 by τ_1, τ_2, we see that the order is reduced to p and q, respectively. They are inverse to the rose curves $\rho'=1/\rho$, including as special cases many famous curves, and have been extensively studied under the names Cotes spirals, epi or Ähren curves.* We note a few properties.

$p>q$. The homogeneous point equations, after removing the factor $(t_1t_2)^q$, reduce to

$$x_1 = 2at_1^{p+q}t_2^{p-q}, \qquad x_2 = 2at_1^{p-q}t_2^{p+q}, \qquad x_3 = t_1^{2p} + t_2^{2p} \tag{12}$$

and the line equations, factoring out $(t_1t_2)^{p-q-1}$, to

$$\begin{aligned} u_1 &= t_2^{2q}[(p+q)t_1^{2p} - (p-q)t_2^{2p}], \\ u_2 &= t_1^{2q}[-(p-q)t_1^{2p} + (p+q)t_2^{2p}], \\ u_3 &= -4aq(t_1t_2)^{p+q}. \end{aligned} \tag{13}$$

The flex form, neglecting a numerical factor, becomes

$$(t_1t_2)^{2p-3}(t_1^{2p} + t_2^{2p}). \tag{14}$$

The (isolated) multiple point at the origin is a $(2p-2q)$-fold point with just two branches, formed by two $(p-q)$-fold points each with coincident parameters, falling together. The point absorbs $2p-2q-2$ cusps, since the class is reduced by this number, and $2p+4q-2$ additional flexes.

We observe that the section by the x_3-axis is a factor of the flex form and we infer

The p nodes at infinity are biflecnodes.

The curve admits a group of order $4p$, having $2p$ axes of symmetry, and thus enjoys the maximum symmetry for an algebraic curve of order $2p$.§ The biflecnodes are centers of reflexion, lying on the axes $x^p+\bar{x}^p=0$ when q is odd

* For references, see G. Loria, Spezielle Algebraische und Transzendente Ebene Kurven, Leipzig, 1902, p. 367.

§ R. M. Winger, Some applications of groups to geometry, this MONTHLY, vol. 37, 1930, p. 4, where through a slip the maximum number of symmetry axes is given as $2n$ instead of n.

but on the complementary set $x^p-\bar{x}^p=0$ when q is even. There are $2p$ real asymptotes, the biflecnodal tangents, parallel in pairs to the axes on which the nodes lie.

$p<q$. The point equations now are

$$x_1 = 2at_1^{2q}, \qquad x_2 = 2at_2^{2q}, \qquad x_3 = (t_1t_2)^{q-p}(t_1^{2p} + t_2^{2p}). \tag{15}$$

The line equations have the same form as (13) but are really not the same because of the different hypotheses regarding p and q.

The flex form, except for a numerical factor, is

$$(t_1t_2)^{3q-p-3}(t_1^{2p} + t_2^{2p}), \tag{16}$$

so that the nodes at infinity are again biflecnodes.

There is a $(q-p)$-fold point at I and at J with coincident parameters and a single tangent with $2p$-point intersection, each absorbing $q-p-1$ cusps and $q+p-1$ additional flexes.

Again the group is of order $4p$, containing $2p$ reflexions, but the symmetry is less than maximum since $p<q$. The biflecnodes and the $2p$ real asymptotes are disposed as in the previous case.

We append some graphs for lower values of p and q.

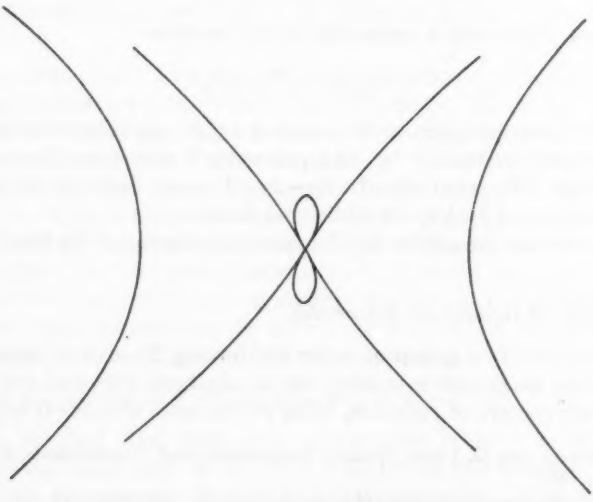

FIG. 1: $p=2$, $q=1$, $k=2a$.

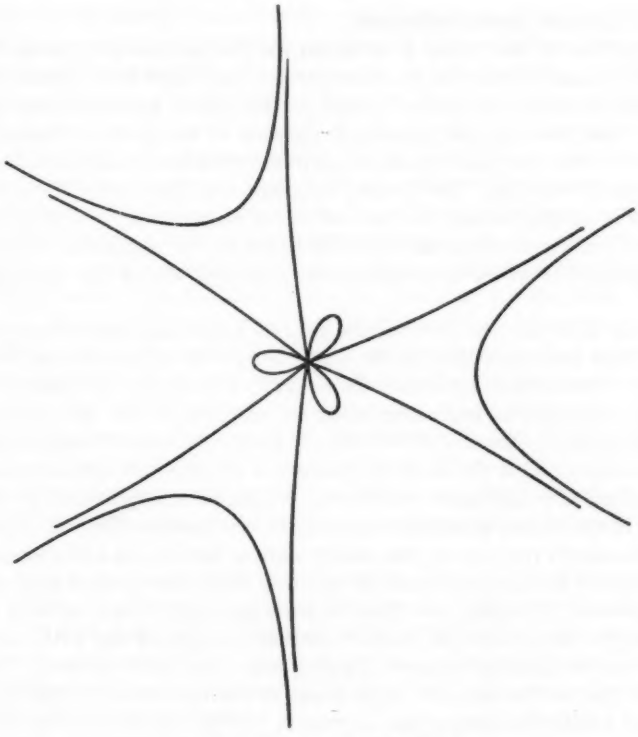

FIG. 2: $p=3$, $q=1$, $k=2a$.

LEONHARD EULER'S INTEGRAL: A HISTORICAL PROFILE OF THE GAMMA FUNCTION

IN MEMORIAM: MILTON ABRAMOWITZ

PHILIP J. DAVIS, National Bureau of Standards, Washington, D. C.

Many people think that mathematical ideas are static. They think that the ideas originated at some time in the historical past and remain unchanged for all future times. There are good reasons for such a feeling. After all, the formula for the area of a circle was πr^2 in Euclid's day and at the present time is still πr^2. But to one who knows mathematics from the inside, the subject has rather the feeling of a living thing. It grows daily by the accretion of new information, it changes daily by regarding itself and the world from new vantage points, it maintains a regulatory balance by consigning to the oblivion of irrelevancy a

fraction of its past accomplishments.

The purpose of this essay is to illustrate this process of growth. We select one mathematical object, the gamma function, and show how it grew in concept and in content from the time of Euler to the recent mathematical treatise of Bourbaki, and how, in this growth, it partook of the general development of mathematics over the past two and a quarter centuries. Of the so-called "higher mathematical functions," the gamma function is undoubtedly the most fundamental. It is simple enough for juniors in college to meet but deep enough to have called forth contributions from the finest mathematicians. And it is sufficiently compact to allow its profile to be sketched within the space of a brief essay.

The year 1729 saw the birth of the gamma function in a correspondence between a Swiss mathematician in St. Petersburg and a German mathematician in Moscow. The former: Leonhard Euler (1707–1783), then 22 years of age, but to become a prodigious mathematician, the greatest of the 18th century. The latter: Christian Goldbach (1690–1764), a savant, a man of many talents and in correspondence with the leading thinkers of the day. As a mathematician he was something of a dilettante, yet he was a man who bequeathed to the future a problem in the theory of numbers so easy to state and so difficult to prove that even to this day it remains on the mathematical horizon as a challenge.

The birth of the gamma function was due to the merging of several mathematical streams. The first was that of interpolation theory, a very practical subject largely the product of English mathematicians of the 17th century but which all mathematicians enjoyed dipping into from time to time. The second stream was that of the integral calculus and of the systematic building up of the formulas of indefinite integration, a process which had been going on steadily for many years. A certain ostensibly simple problem of interpolation arose and was bandied about unsuccessfully by Goldbach and by Daniel Bernoulli (1700–1784) and even earlier by James Stirling (1692–1770). The problem was posed to Euler. Euler announced his solution to Goldbach in two letters which were to be the beginning of an extensive correspondence which lasted the duration of Goldbach's life. The first letter dated October 13, 1729 dealt with the interpolation problem, while the second dated January 8, 1730 dealt with integration and tied the two together. Euler wrote Goldbach the merest outline, but within a year he published all the details in an article *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt*. This article can now be found reprinted in Volume I_{14} of Euler's *Opera Omnia*.

Since the interpolation problem is the easier one, let us begin with it. One of the simplest sequences of integers which leads to an interesting theory is 1, $1+2$, $1+2+3$, $1+2+3+4$, $\cdots$. These are the triangular numbers, so called because they represent the number of objects which can be placed in a triangular array of various sizes. Call the nth one T_n. There is a formula for T_n which is learned in school algebra: $T_n=\frac{1}{2}n(n+1)$.

What, precisely, does this formula accomplish? In the first place, it simplifies

computation by reducing a large number of additions to three fixed operations: one of addition, one of multiplication, and one of division. Thus, instead of adding the first hundred integers to obtain T_{100}, we can compute $T_{100}=\frac{1}{2}(100)(100+1)=5050$. Secondly, even though it doesn't make literal sense to ask for, say, the sum of the first $5\frac{1}{2}$ integers, the formula for T_n produces an answer to this. For whatever it is worth, the formula yields $T_{5\frac{1}{2}}=\frac{1}{2}(5\frac{1}{2})(5\frac{1}{2}+1)=17\frac{7}{8}$. In this way, the formula extends the scope of the original problem to values of the variable other than those for which it was originally defined and solves the problem of interpolating between the known elementary values.

This type of question, one which asks for an extension of meaning, cropped up frequently in the 17th and 18th centuries. Consider, for instance, the algebra of exponents. The quantity a^m is defined initially as the product of m successive a's. This definition has meaning when m is a positive integer, but what would $a^{5\frac{1}{2}}$ be? The product of $5\frac{1}{2}$ successive a's? The mysterious definitions $a^0=1$, $a^{m/n}=\sqrt[n]{a^m}$, $a^{-m}=1/a^m$ which solve this enigma and which are employed so fruitfully in algebra were written down explicitly for the first time by Newton in 1676. They are justified by a utility which derives from the fact that the definition leads to continuous exponential functions and that the law of exponents $a^m\cdot a^n=a^{m+n}$ becomes meaningful for all exponents whether positive integers or not.

Other problems of this type proved harder. Thus, Leibnitz introduced the notation d^n for the nth iterate of the operation of differentiation. Moreover, he identified d^{-1} with $\int$ and d^{-n} with the iterated integral. Then he tried to breathe some sense into the symbol d^n when n is any real value whatever. What, indeed, is the $5\frac{1}{2}$th derivative of a function? This question had to wait almost two centuries for a satisfactory answer.

THE FACTORIALS

n:	1	2	3	4	5	6	7	8	···
$n!$:	1	2	6	24	120	720	5040	40,320	···

FIG. 1

INTELLIGENCE TEST

Question: What number should be inserted in the lower line half way between the upper 5 and 6?

Euler's Answer: 287.8852 · · · . Hadamard's Answer: 280.3002 · · · .

But to return to our sequence of triangular numbers. If we change the plus signs to multiplication signs we obtain a new sequence: $1, 1\cdot 2, 1\cdot 2\cdot 3, \cdots$. This is the sequence of factorials. The factorials are usually abbreviated $1!, 2!, 3!, \cdots$ and the first five are 1, 2, 6, 24, 120. They grow in size very rapidly. The number 100! if written out in full would have 158 digits. By contrast, $T_{100}=5050$ has a

mere four digits. Factorials are omnipresent in mathematics; one can hardly open a page of mathematical analysis without finding it strewn with them. This being the case, is it possible to obtain an easy formula for computing the factorials? And is it possible to interpolate between the factorials? What should $5\frac{1}{2}!$ be? (See Fig. 1.) This is the interpolation problem which led to the gamma function, the interpolation problem of Stirling, of Bernoulli, and of Goldbach. As we know, these two problems are related, for when one has a formula there is the possibility of inserting intermediate values into it. And now comes the surprising thing. There is no, in fact there can be, no formula for the factorials which is of the simple type found for T_n. This is implicit in the very title Euler chose for his article. Translate the Latin and we have *On transcendental progressions whose general term cannot be expressed algebraically.* The solution to factorial interpolation lay deeper than "mere algebra." Infinite processes were required.

In order to appreciate a little better the problem confronting Euler it is useful to skip ahead a bit and formulate it in an up-to-date fashion: find a reasonably simple function which at the integers 1, 2, 3, $\cdots$ takes on the factorial values 1, 2, 6, $\cdots$. Now today, a function is a relationship between two sets of numbers wherein to a number of one set is assigned a number of the second set. What is stressed is the relationship and not the nature of the rules which serve to determine the relationship. To help students visualize the function concept in its full generality, mathematics instructors are accustomed to draw a curve full of twists and discontinuities. The more of these the more general the function is supposed to be. Given, then, the points (1,1), (2, 2), (3, 6), (4, 24), $\cdots$ and adopting the point of view wherein "function" is what we have just said, the problem of interpolation is one of finding a curve which passes through the given points. This is ridiculously easy to solve. It can be done in an unlimited number of ways. Merely take a pencil and draw some curve—any curve will do—which passes through the points. Such a curve automatically defines a function which solves the interpolation problem. In this way, too free an attitude as to what constitutes a function solves the problem trivially and would enrich mathematics but little. Euler's task was different. In the early 18th century, a function was more or less synonymous with a formula, and by a formula was meant an expression which could be derived from elementary manipulations with addition, subtraction, multiplication, division, powers, roots, exponentials, logarithms, differentiation, integration, infinite series, *i.e.*, one which came from the ordinary processes of mathematical analysis. Such a formula was called an *expressio analytica*, an analytical expression. Euler's task was to find, if he could, an analytical expression arising naturally from the corpus of mathematics which would yield factorials when a positive integer was inserted, but which would still be meaningful for other values of the variable.

It is difficult to chronicle the exact course of scientific discovery. This is particularly true in mathematics where one traditionally omits from articles and books all accounts of false starts, of the initial years of bungling, and where one may develop one's topic forward or backward or sideways in order to heighten

the dramatic effect. As one distinguished mathematician put it, a mathematical result must appear straight from the heavens as a *deus ex machina* for students to verify and accept but not to comprehend. Apparently, Euler, experimenting with infinite products of numbers, chanced to notice that if n is a positive integer,

$$(1) \qquad \left[\left(\frac{2}{1}\right)^n \frac{1}{n+1}\right]\left[\left(\frac{3}{2}\right)^n \frac{2}{n+2}\right]\left[\left(\frac{4}{3}\right)^n \frac{3}{n+3}\right]\cdots = n!.$$

Leaving aside all delicate questions as to the convergence of the infinite product, the reader can verify this equation by cancelling out all the common factors which appear in the top and bottom of the left-hand side. Moreover, the left-hand side is defined (at least formally) for all kinds of n other than negative integers. Euler noticed also that when the value $n=\frac{1}{2}$ is inserted, the left-hand side yields (after a bit of manipulation) the famous infinite product of the Englishman John Wallis (1616–1703):

$$(2) \qquad \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\left(\frac{8\cdot 8}{7\cdot 9}\right)\cdots = \pi/2.$$

With this discovery Euler could have stopped. His problem was solved. Indeed, the whole theory of the gamma function can be based on the infinite product (1) which today is written more conventionally as

$$(3) \qquad \lim_{m\to\infty} \frac{m!(m+1)^n}{(n+1)(n+2)\cdots(n+m)}.$$

However, he went on. He observed that his product displayed the following curious phenomenon: for some values of n, namely integers, it yielded integers, whereas for another value, namely $n=\frac{1}{2}$, it yielded an expression involving π. Now π meant circles and their quadrature, and quadratures meant integrals, and he was familiar with integrals which exhibited the same phenomenon. It therefore occurred to him to look for a transformation which would allow him to express his product as an integral.

He took up the integral $\int_0^1 x^e(1-x)^n dx$. Special cases of it had already been discussed by Wallis, by Newton, and by Stirling. It was a troublesome integral to handle, for the indefinite integral is not always an elementary function of x. Assuming that n is an integer, but that e is an arbitrary value, Euler expanded $(1-x)^n$ by the binomial theorem, and without difficulty found that

$$(4) \qquad \int_0^1 x^e(1-x)^n dx = \frac{1\cdot 2\cdots n}{(e+1)(e+2)\cdots(e+n+1)}.$$

Euler's idea was now to isolate the $1\cdot 2\cdots n$ from the denominator so that he would have an expression for $n!$ as an integral. He proceeds in this way. (Here we follow Euler's own formulation and nomenclature, marking with an * those

formulas which occur in the original paper. Euler wrote a plain $\int$ for $\int_0^1$.) He substituted f/g for e and found

$$(5) \qquad \int_0^1 x^{f/g}(1-x)^n dx = \frac{g^{n+1}}{f+(n+1)g(f+g)(f+2\cdot g)\cdots(f+n\cdot g)}.$$

And so,

$$(6)^* \qquad \frac{1\cdot 2\cdots n}{(f+g)(f+2\cdot g)\cdots(f+n\cdot g)} = \frac{f+(n+1)g}{g^{n+1}}\int x^{f/g}dx(1-x)^n.$$

He observed that he could isolate the $1\cdot 2\cdots n$ if he set $f=1$ and $g=0$ in the left-hand member, but that if he did so, he would obtain on the right an indeterminate form which he writes quaintly as

$$(7)^* \qquad \int \frac{x^{1/0}dx(1-x)^n}{0^{n+1}}.$$

He now proceeded to find the value of the expression (7)*. He first made the substitution $x^{g/(f+g)}$ in place of x. This gave him

$$(8)^* \qquad \frac{g}{f+g}x^{-f/(g+f)}dx$$

in place of dx and hence, the right-hand member of (6)* becomes

$$(9)^* \qquad \frac{f+(n+1)g}{g^{n+1}}\int \frac{g}{f+g}dx(1-x^{g/(f+g)})^n.$$

Once again, Euler made a trial setting of $f=1$, $g=0$ having presumably reduced this integral first to

$$(10) \qquad \frac{f+(n+1)g}{(f+g)^{n+1}}\int_0^1\left(\frac{1-x^{g/(f+g)}}{g/(f+g)}\right)^n dx,$$

and this yielded the indeterminate

$$(11)^* \qquad \int dx\frac{(1-x^0)^n}{0^n}.$$

He now considered the related expression $(1-x^z)/z$, for vanishing z. He differentiated the numerator and denominator, as he says, by a known (l'Hospital's) rule and obtained

$$(12)^* \qquad \frac{-x^z dz lx}{dz} \qquad (lx = \log x),$$

which for $z=0$ produced $-lx$. Thus,

$$(13)^* \qquad (1-x^0)/0 = -lx$$

and

$$(14)^* \qquad (1 - x^0)^n/0^n = (-lx)^n.$$

He therefore concluded that

$$(15) \qquad n! = \int_0^1 (-\log x)^n dx.$$

This gave him what he wanted, an expression for $n!$ as an integral wherein values other than positive integers may be substituted. The reader is encouraged to formulate his own criticism of Euler's derivation.

Students in advanced calculus generally meet Euler's integral first in the form

$$(16) \qquad \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \qquad e = 2.71828 \cdots.$$

This modification of the integral (15) as well as the Greek Γ is due to Adrien Marie Legendre (1752–1833). Legendre calls the integral (4) with which Euler started his derivation the first Eulerian integral and (15) the second Eulerian integral. The first Eulerian integral is currently known as the Beta function and is now conventionally written

$$(17) \qquad B(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1}dx.$$

With the tools available in advanced calculus, it is readily established (how easily the great achievements of the past seem to be comprehended and duplicated!) that the integral possesses meaning when $x>0$ and thus yields a certain function $\Gamma(x)$ defined for these values. Moreover,

$$(18) \qquad \Gamma(n + 1) = n!$$

whenever n is a positive integer.* It is further established that for all $x>0$

$$(19) \qquad x\Gamma(x) = \Gamma(x + 1).$$

This is the so-called recurrence relation for the gamma function and in the years following Euler it plays, as we shall see, an increasingly important role in its theory. These facts, plus perhaps the relationship between Euler's two types of integrals

$$(20) \qquad B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m + n)$$

and the all important Stirling formula

* Legendre's notation shifts the argument. Gauss introduced a notation $\pi(x)$ free of this defect. Legendre's notation won out, but continues to plague many people. The notations Γ, π, and ! can all be found today.

(21) $$\Gamma(x) \sim e^{-x}x^{x-1/2}\sqrt{(2\pi)},$$

which gives us a relatively simple approximate expression for $\Gamma(x)$ when x is large, are about all that advanced calculus students learn of the gamma function. Chronologically speaking, this puts them at about the year 1750. The play has hardly begun.

Just as the simple desire to extend factorials to values in between the integers led to the discovery of the gamma function, the desire to extend it to negative values and to complex values led to its further development and to a more profound interpretation. Naive questioning, uninhibited play with symbols may have been at the very bottom of it. What is the value of $(-5\frac{1}{2})!$? What is the value of $\sqrt{(-1)}!$? In the early years of the 19th century, the action broadened and moved into the complex plane (the set of all numbers of the form $x+iy$, where $i=\sqrt{(-1)}$) and there it became part of the general development of the theory of functions of a complex variable that was to form one of the major chapters in mathematics. The move to the complex plane was initiated by Karl Friedrich Gauss (1777–1855), who began with Euler's product as his starting point. Many famous names are now involved and not just one stage of action but many stages. It would take too long to record and describe each forward step taken. We shall have to be content with a broader picture.

Three important facts were now known: Euler's integral, Euler's product, and the functional or recurrence relationship $x\Gamma(x)=\Gamma(x+1)$, $x>0$. This last is the generalization of the obvious arithmetic fact that for positive integers, $(n+1)n!=(n+1)!$ It is a particularly useful relationship inasmuch as it enables us by applying it over and over again to reduce the problem of evaluating a factorial of an arbitrary real number whole or otherwise to the problem of evaluating the factorial of an appropriate number lying between 0 and 1. Thus, if we write $n=4\frac{1}{2}$ in the above formula we obtain $(4\frac{1}{2}+1)!=5\frac{1}{2}(4\frac{1}{2})!$ If we could only find out what $(4\frac{1}{2})!$ is, then we would know that $(5\frac{1}{2})!$ is. This process of reduction to lower numbers can be kept up and yields

(22) $$(5\tfrac{1}{2})! = (3/2)(5/2)(7/2)(9/2)(11/2)(1/2)!$$

and since we have $(\frac{1}{2})!=\frac{1}{2}\sqrt{\pi}$ from (1) and (2), we can now compute our answer. Such a device is obviously very important for anyone who must do calculations with the gamma function. Other information is forthcoming from the recurrence relationship. Though the formula $(n+1)n!=(n+1)!$ as a condensation of the arithmetic identity $(n+1)\cdot 1\cdot 2\cdot\cdots\cdot n=1\cdot 2\cdot\cdots\cdot n\cdot(n+1)$ makes sense only for $n=1, 2$, etc., blind insertions of other values produce interesting things. Thus, inserting $n=0$, we obtain $0!=1$. Inserting successively $n=-5\frac{1}{2}$, $n=-4\frac{1}{2}$, $\cdots$ and reducing upwards, we discover

(23) $$(-5\tfrac{1}{2})! = (2/1)(-2/1)(-2/3)(-2/5)(-2/7)(-2/9)(1/2)!$$

Since we already know what $(\frac{1}{2})!$ is, we can compute $(-5\frac{1}{2})!$ In this way the recurrence relationship enables us to compute the values of factorials of negative

numbers.

Turning now to Euler's integral, it can be shown that for values of the variable less than 0, the usual theorems of analysis do not suffice to assign a meaning to the integral, for it is divergent. On the other hand, it is meaningful and yields a value if one substitutes for x any complex number of the form $a+bi$ where $a>0$. With such substitutions the integral therefore yields a complex-valued function which is defined for all complex numbers in the right-half of the complex plane and which coincides with the ordinary gamma function for real values. Euler's product is even stronger. With the exception of $0, -1, -2, \cdots$ any complex number whatever can be inserted for the variable and the infinite product will converge, yielding a value. And so it appears that we have at our disposal a number of methods, conceptually and operationally different for extending the domain of definition of the gamma function. Do these different methods yield the same result? They do. But why?

The answer is to be found in the notion of an analytic function. This is the focal point of the theory of functions of a complex variable and an outgrowth of the older notion of an analytical expression. As we have hinted, earlier mathematics was vague about this notion, meaning by it a function which arose in a natural way in mathematical analysis. When later it was discovered by J. B. J. Fourier (1768–1830) that functions of wide generality and functions with unpleasant characteristics could be produced by the infinite superposition of ordinary sines and cosines, it became clear that the criterion of "arising in a natural way" would have to be dropped. The discovery simultaneously forced a broadening of the idea of a function and a narrowing of what was meant by an analytic function.

Analytic functions are not so arbitrary in their behavior. On the contrary, they possess strong internal ties. Defined very precisely as functions which possess a complex derivative or equivalently as functions which possess power series expansions $a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$ they exhibit the remarkable phenomenon of "action at a distance." This means that the behavior of an analytic function over any interval no matter how small is sufficient to determine completely its behavior everywhere else; its potential range of definition and its values are theoretically obtainable from this information. Analytic functions, moreover, obey the principle of the permanence of functional relationships; if an analytic function satisfies in some portions of its region of definition a certain functional relationship, then it must do so wherever it is defined. Conversely, such a relationship may be employed to extend its definition to unknown regions. Our understanding of the process of analytic continuation, as this phenomenon is known, is based upon the work of Bernhard Riemann (1826–1866) and Karl Weierstrass (1815–1897). The complex-valued function which results from the substitution of complex numbers into Euler's integral is an analytic function. The function which emerges from Euler's product is an analytic function. The recurrence relationship for the gamma function if satisfied in some region must be satisfied in any other region to which the function

can be "continued" analytically and indeed may be employed to effect such extensions. All portions of the complex plane, with the exception of the values $0, -1, -2, \cdots$ are accessible to the complex gamma function which has become the unique, analytic extension to complex values of Euler's integral (see Fig. 3).

THE GAMMA FUNCTION

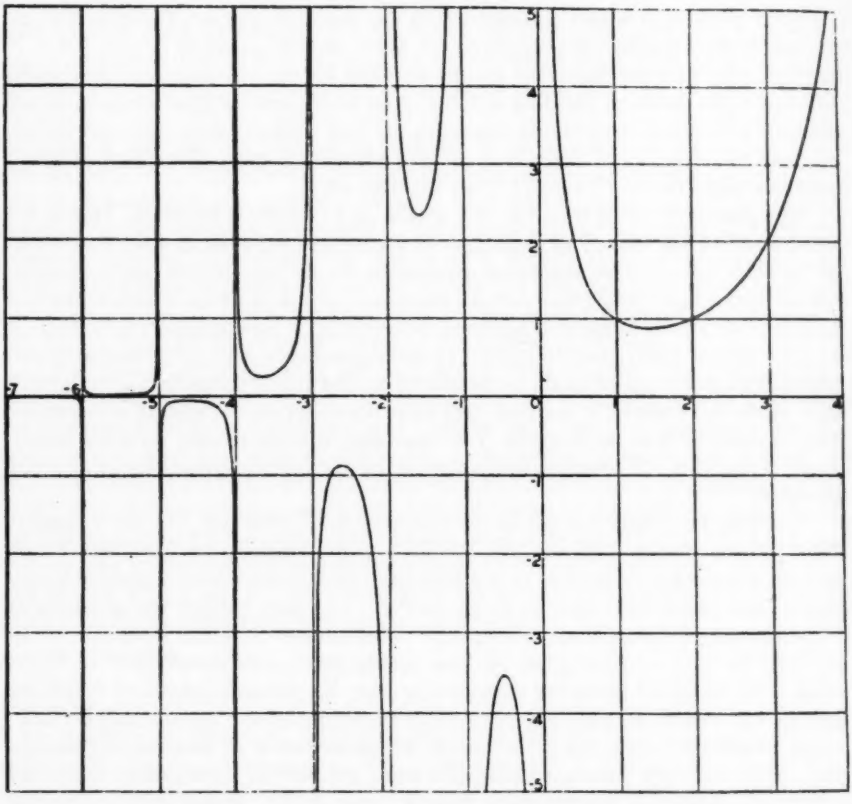

FIG. 2*

To understand why there should be excluded points observe that $\Gamma(x) = \Gamma(x+1)/x$, and as x approaches 0, we obtain $\Gamma(0) = 1/0$. This is $+\infty$ or $-\infty$ depending whether 0 is approached through positive or negative values. The

* From: H. T. Davis, Tables of the Higher Mathematical Functions, vol. I, Bloomington, Indiana, 1933.

functional equation (19) then, induces this behavior over and over again at each of the negative integers. The (real) gamma function is comprised of an infinite number of disconnected portions opening up and down alternately. The portions corresponding to negative values are each squeezed into an infinite strip one unit in width, but the major portion which corresponds to positive x and which contains the factorials is of infinite width (see Fig. 2). Thus, there are excluded points for the gamma function at which it exhibits from the ordinary (real variable) point of view a somewhat unpleasant and capricious behavior.

THE ABSOLUTE VALUE OF THE COMPLEX GAMMA FUNCTION, EXHIBITING THE POLES AT THE NEGATIVE INTEGERS

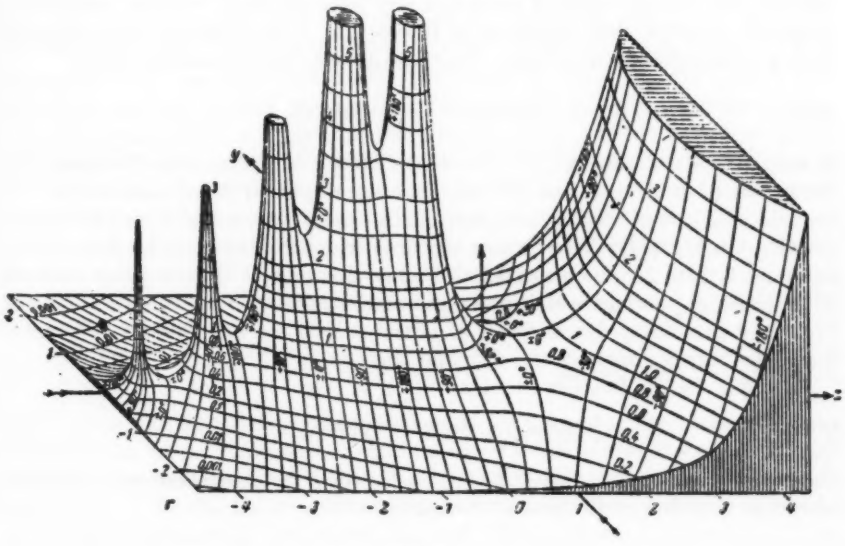

FIG. 3*

But from the complex point of view, these points of singular behavior (singular in the sense of Sherlock Holmes) merit special study and become an important part of the story. In pictures of the complex gamma function they show up as an infinite row of "stalagmites," each of infinite height (the ones in the figure are truncated out of necessity) which become more and more needlelike as they go out to infinity (see Fig. 3). They are known as poles. Poles are points where the function has an infinite behavior of especially simple type, a behavior which is akin to that of such simple functions as the hyperbola $y=1/x$ at $x=0$ or of $y=\tan x$ at $x=\pi/2$. The theory of analytic functions is especially interested

* From: E. Jahnke and F. Emde, Tafeln höherer Funktionen, 4th ed., Leipzig, 1948.

in singular behavior, and devotes much space to the study of the singularities. Analytic functions possess many types of singularity but those with only poles are known as meromorphic. There are also functions which are lucky enough to possess no singularities for finite arguments. Such functions form an elite and are known as entire functions. They are akin to polynomials while the meromorphic functions are akin to the ratio of polynomials. The gamma function is meromorphic. Its reciprocal, $1/\Gamma(x)$, has on the contrary no excluded points. There is no trouble anywhere. At the points $0, -1, -2, \cdots$ it merely becomes zero. And the zero value which occurs an infinity of times, is strongly reminiscent of the sine.

In the wake of the extension to the complex many remarkable identities emerge, and though some of them can and were obtained without reference to complex variables, they acquire a far deeper and richer meaning when regarded from the extended point of view. There is the reflection formula of Euler

$$\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z. \tag{24}$$

It is readily shown, using the recurrence relation of the gamma function, that the product $\Gamma(z)\Gamma(1-z)$ is a periodic function of period 2; but despite the fact that $\sin \pi z$ is one of the simplest periodic functions, who could have anticipated the relationship (24)? What, after all, does trigonometry have to do with the sequence 1, 2, 6, 24 which started the whole discussion? Here is a fine example of the delicate patterns which make the mathematics of the period so magical. From the complex point of view, a partial reason for the identity lies in the similarity between zeros of the sine and the poles of the gamma function.

There is the duplication formula

$$\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \tag{25}$$

discovered by Legendre and extended by Gauss in his researches on the hypergeometric function to the multiplication formula

$$\Gamma(nz) = (2\pi)^{1/2(1-n)}n^{nz-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdot\cdots\cdot\Gamma\left(\frac{z+n-1}{n}\right). \tag{26}$$

There are pretty formulas for the derivatives of the gamma function such as

$$d^2 \log \Gamma(z)/dz^2 = \frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \cdots. \tag{27}$$

This is an example of a type of infinite series out of which G. Mittag-Leffler (1846–1927) later created his theory of partial fraction developments of meromorphic functions. There is the intimate relationship between the gamma function and the zeta function which has been of fundamental importance in studying the distribution of the prime numbers,

$$\zeta(z) = \zeta(1-z)\Gamma(1-z)2^z\pi^{z-1}\sin\tfrac{1}{2}\pi z, \tag{28}$$

where

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots. \tag{29}$$

This formula has some interesting history related to it. It was first proved by Riemann in 1859 and was conventionally attributed to him. Yet in 1894 it was discovered that a modified version of the identity appears in some work of Euler which had been done in 1749. Euler did not claim to have proved the formula. However, he "verified" it for integers, for $\frac{1}{2}$, and for 3/2. The verification for $\frac{1}{2}$ is by direct substitution, but for all the other values, Euler works with divergent infinite series. This was more than 100 years in advance of a firm theory of such series, but with unerring intuition, he proceeded to sum them by what is now called the method of Abel summation. The case 3/2 is even more interesting. There, invoking both divergent series and numerical evaluation, he came out with numerical agreement to 5 decimal places! All this work convinced him of the truth of his identity. Rigorous modern proofs do not require the theory of divergent series, but the notions of analytic continuation are crucial.

In view of the essential unity of the gamma function over the whole complex plane it is theoretically and aesthetically important to have a formula which works for all complex numbers. One such formula was supplied in 1848 by F. W. Newman:

$$1/\Gamma(z) = ze^{\gamma z}\{(1+z)e^{-z}\}\{(1+z/2)e^{-z/2}\}\cdots, \text{ where } \gamma = .57721\ 56649\cdots. \tag{30}$$

This formula is essentially a factorization of $1/\Gamma(z)$ and is much the same as a factorization of polynomials. It exhibits clearly where the function vanishes. Setting each factor equal to zero we find that $1/\Gamma(z)$ is zero for $z=0$, $z=-1$, $z=-2$, $\cdots$. In the hands of Weierstrass, it became the starting point of his particular discussion of the gamma function. Weierstrass was interested in how functions other than polynomials may be factored. A number of isolated factorizations were then known. Newman's formula (30) and the older factorization of the sine

$$\sin\pi z = \pi z(1-z^2)\left(1-\frac{z^2}{4}\right)\left(1-\frac{z^2}{9}\right)\cdots \tag{31}$$

are among them. The factorization of polynomials is largely an algebraic matter but the extension to functions such as the sine which have an infinity of roots required the systematic building up of a theory of infinite products. In 1876 Weierstrass succeeded in producing an extensive theory of factorizations which included as special cases these well-known infinite products, as well as certain doubly periodic functions.

In addition to showing the roots of $1/\Gamma(z)$, formula (30) does much more.

It shows immediately that the reciprocal of the gamma function is a much less difficult function to deal with than the gamma function itself. It is an entire function, that is, one of those distinguished functions which possesses no singularities whatever for finite arguments. Weierstrass was so struck by the advantages to be gained by starting with $1/\Gamma(z)$ that he introduced a special notation for it. He called $1/\Gamma(u+1)$ the *factorielle* of u and wrote $Fc(u)$.

The theory of functions of a complex variable unifies a hotch-potch of curves and a patchwork of methods. Within this theory, with its highly developed studies of infinite series of various types, was brought to fruition Stirling's unsuccessful attempts at solving the interpolation problem for the factorials. Stirling had done considerable work with infinite series of the form

$$A+Bz+Cz(z-1)+Dz(z-1)(z-2)+\cdots.$$

This series is particularly useful for fitting polynomials to values given at the integers $z=0, 1, 2, \cdots$. The method of finding the coefficients $A, B, C, \cdots$ was well known. But when an infinite amount of fitting is required, much more than simple formal work is needed, for we are then dealing with a bona fide infinite series whose convergence must be investigated. Starting from the series $1, 2, 6, 24, \cdots$, Stirling found interpolating polynomials via the above series. The resultant infinite series is divergent. The factorials grow too rapidly in size. Stirling realized this and put out the suggestion that if perhaps one started with the logarithms of the factorials instead of the factorials themselves the size might be cut down sufficiently for one to do something. There the matter rested until 1900 when Charles Hermite (1822–1901) wrote down the Stirling series for $\log \Gamma(1+z)$:

$$\log \Gamma(1+z) = \frac{z(z-1)}{1\cdot 2}\log 2 + \frac{z(z-1)(z-2)}{1\cdot 2\cdot 3}(\log 3 - 2\log 2) + \cdots \tag{32}$$

and showed that this identity is valid whenever z is a complex number of the form $a+ib$ with $a>0$. The identity itself could have been written down by Stirling, but the proof would have been another matter. An even simpler starting point is the function $\psi(z)=(d/dz)\log\Gamma(z)$, now known as the digamma or psi function. This leads to the Stirling series

$$\frac{d}{dz}\log\Gamma(z) = -\gamma + (z-1) - \frac{(z-1)(z-2)}{2\cdot 2!} + \frac{(z-1)(z-2)(z-3)}{3\cdot 3!}\cdots, \tag{33}$$

which in 1847 was proved convergent for $a>0$ by M. A. Stern, a teacher of Riemann. All these matters are today special cases of the extensive theory of the convergence of interpolation series.

Functions are the building blocks of mathematical analysis. In the 18th and 19th centuries mathematicians devoted much time and loving care to develop-

ing the properties and interrelationships between special functions. Powers, roots, algebraic functions, trigonometric functions, exponential functions, logarithmic functions, the gamma function, the beta function, the hypergeometric function, the elliptic functions, the theta function, the Bessel function, the Matheiu function, the Weber function, Struve function, the Airy function, Lamé functions, literally hundreds of special functions were singled out for scrutiny and their main features were drawn. This is an art which is not much cultivated these days. Times have changed and emphasis has shifted. Mathematicians on the whole prefer more abstract fare. Large classes of functions are studied instead of individual ones. Sociology has replaced biography. The field of special functions, as it is now known, is left largely to a small but ardent group of enthusiasts plus those whose work in physics or engineering confronts them directly with the necessity of dealing with such matters.

The early 1950's saw the publication of some very extensive computations of the gamma function in the complex plane. Led off in 1950 by a six-place table computed in England, it was followed in Russia by the publication of a very extensive six-place table. This in turn was followed in 1954 by the publication by the National Bureau of Standards in Washington of a twelve-place table. Other publications of the complex gamma function and related functions have appeared in this country, in England, and in Japan. In the past, the major computations of the gamma function had been confined to real values. Two fine tables, one by Gauss in 1813 and one by Legendre in 1825, seemed to answer the mathematical needs of a century. Modern technology had also caught up with the gamma function. The tables of the 1800's were computed laboriously by hand, and the recent ones by electronic digital computers.

But what touched off this spate of computational activity? Until the initial labors of H. T. Davis of Indiana University in the early 1930's, the complex values of the gamma function had hardly been touched. It was one of those curious turns of events wherein the complex gamma function appeared in the solution of various theoretical problems of atomic and nuclear theory. For instance, the radial wave functions for positive energy states in a Coulomb field leads to a differential equation whose solution involves the complex gamma function. The complex gamma function enters into formulas for the scattering of charged particles, for the nuclear forces between protons, in Fermi's approximate formula for the probability of β-radiation, and in many other places. The importance of these problems to physicists has had the side effect of computational mathematics finally catching up with two and a quarter centuries of theoretical development.

As analysis grew, both creating special functions and delineating wide classes of functions, various classifications were used in order to organize them for purposes of convenient study. The earlier mathematicians organized functions from without, operationally, asking what operations of arithmetic or calculus had to be performed in order to achieve them. Today, there is a much greater tendency to look at functions from within, organically, considering their construction as

achieved and asking what geometrical characteristics they possess. In the earlier classification we have at the lowest and most accessible level, powers, roots, and all that could be concocted from them by ordinary algebraic manipulation. These came to be known as algebraic functions. The calculus, with its characteristic operation of taking limits, introduced logarithms and exponentials, the latter encompassing, as Euler showed, the sines and cosines of trigonometry which had been available from earlier periods of discovery. There is an impassable wall between the algebraic functions and the new limit-derived ones. This wall consists in the fact that try as one might to construct, say, a trigonometric function out of the finite material of algebra, one cannot succeed. In more technical language, the algebraic functions are closed with respect to the processes of algebra, and the trigonometric functions are forever beyond its pale. (By way of a simple analogy: the even integers are closed with respect to the operations of addition, subtraction, and multiplication; you cannot produce an odd integer from the set of even integers using these tools.) This led to the concept of transcendental functions. These are functions which are not algebraic. The transcendental functions count among their members, the trigonometric functions, the logarithms, the exponentials, the elliptic functions, in short, practically all the special functions which had been singled out for special study. But such an indescriminate dumping produced too large a class to handle. The transcendentals had to be split further for convenience. A major tool of analysis is the differential equation, expressing the relationship between a function and its rate of growth. It was found that some functions, say the trigonometric functions, although they are transcendental and do not therefore satisfy an algebraic equation, nonetheless satisfy a differential equation whose coefficients are algebraic. The solutions of algebraic differential equations are an extensive though not all-encompassing class of transcendental functions. They count among their members a good many of the special functions which arise in mathematical physics.

Where does the gamma function fit into this? It is not an algebraic function. This was recognized early. It is a transcendental function. But for a long while it was an open question whether the gamma function satisfied an algebraic differential equation. The question was settled negatively in 1887 by O. Hölder (1859–1937). It does not. It is of a higher order of transcendency. It is a so-called transcendentally transcendent function, unreachable by solving algebraic equations, and equally unreachable by solving algebraic differential equations. The subject has interested many people through the years and in 1925 Alexander Ostrowski, now Professor Emeritus of the University of Basel, Switzerland, gave an alternate proof of Hölder's theorem.

Problems of classification are extremely difficult to handle. Consider, for instance, the following: Can the equation x^7+8x+1 be solved with radicals? Is π transcendental? Can $\int dx/\sqrt{(x^3+1)}$ be found in terms of specified elementary functions? Can the differential equation $dy/dx=(1/x)+(1/y)$ be resolved with quadratures? The general problems of which these are representatives are

even today far from solved and this despite famous theories such as Galois Theory, Lie theory, theory of Abelian integrals which have derived from such simple questions. Each individual problem may be a one-shot affair to be solved by individual methods involving incredible ingenuity.

HADAMARD'S FACTORIAL FUNCTION

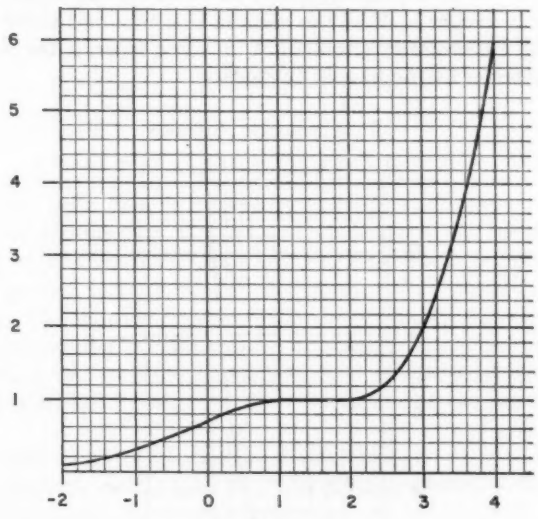

FIG. 4

There are infinitely many functions which produce factorials. The function

$$F(x) = (1/\Gamma(1-x))\,(d/dx)\log\{\Gamma((1-x)/2)/\Gamma(1-x/2)\}$$

is an entire analytic function which coincides with the gamma function at the positive integers. It satisfies the functional equation $F(x+1)=xF(x)+(1/\Gamma(1-x))$.

We return once again to our interpolation problem. We have shown how, strictly speaking, there are an unlimited number of solutions to this problem. To drive this point home, we might mention a curious solution given in 1894 by Jacques Hadamard (1865–). Hadamard found a relatively simple formula involving the gamma function which also produces factorial values at the positive integers. (See Figs. 1 and 4.) But Hadamard's function

$$y = \frac{1}{\Gamma(1-x)}\,\frac{d}{dx}\log\left[\Gamma\left(\frac{1-x}{2}\right)\Big/\,\Gamma\left(1-\frac{x}{2}\right)\right], \tag{34}$$

in strong contrast to the gamma function itself, possesses no singularities anywhere in the finite complex plane. It is an entire analytic solution to the interpolation problem and hence, from the function theoretic point of view, is a simpler solution. In view of all this ambiguity, why then should Euler's solution

be considered the solution par excellence?

From the point of view of integrals, the answer is clear. Euler's integral appears everywhere and is inextricably bound to a host of special functions. Its frequency and simplicity make it fundamental. When the chips are down, it is the very form of the integral and of its modifications which lend it utility and importance. For the interpolatory point of view, we can make no such claim. We must take a deeper look at the gamma function and show that of all the solutions of the interpolation problem, it, in some sense, is the simplest. This is partially a matter of mathematical aesthetics.

A PSEUDOGAMMA FUNCTION

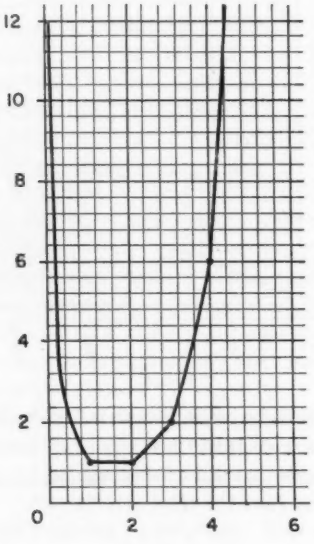

FIG. 5

The function illustrated produces factorials, satisfies the functional equation of the gamma function, and is convex.

We have already observed that Euler's integral satisfies the fundamental recurrence equation, $x\Gamma(x)=\Gamma(x+1)$, and that this equation enables us to compute all the real values of the gamma function from knowledge merely of its values in the interval from 0 to 1. Since the solution to the interpolation problem is not determined uniquely, it makes sense to add to the problem more conditions and to inquire whether the augmented problem then possesses a unique solution. If it does, we will hope that the solution coincides with Euler's. The recurrence relationship is a natural condition to add. If we do so, we find that the gamma function is again not the only function which satisfies this recurrence

relation and produces factorials. One may easily construct a "pseudo" gamma function $\Gamma_S(x)$ by defining it between, say, 1 and 2 in any way at all (subject only to $\Gamma_S(1)=1$, $\Gamma_S(2)=1$), and allowing the recurrence relationship to extend its values everywhere else.

If, for instance, we let $\Gamma_S(x)$ be 1 everywhere between 1 and 2, the recurrence relation leads us to the function (see Fig. 5).

$$
\begin{aligned}
&\Gamma_S(x) = 1/x && 0 < x \leqq 1;\\
&\Gamma_S(x) = 1, && 1 \leqq x \leqq 2;\\
&\Gamma_S(x) = x - 1, && 2 \leqq x \leqq 3;\\
&\Gamma_S(x) = (x-1)(x-2), && 3 \leqq x \leqq 4; \cdots .
\end{aligned} \tag{35}
$$

We might end up with a fairly weird result, depending upon what we start with. Even if we require the final result to be an analytic function, there are ways of doing it. For instance, take any function which is both analytic and periodic with period 1. Call it $p(x)$. Make sure that $p(1)=1$. The function $1+\sin 2\pi x$ will do for $p(x)$. Now multiply the ordinary gamma function $\Gamma(x)$ by $p(x)$ and the result $\Gamma(x)p(x)$ will be a "pseudo" gamma function which is analytic, satisfies the recurrence relation, and produces factorials! Thus, we still do not have enough conditions. We must augment the problem again. But what to add?

By the middle of the 19th century it was recognized that Euler's gamma function was the only continuous function which satisfied simultaneously the recurrence relationship, the reflection formula and the multiplication formula. Weierstrass later showed that the gamma function was the only continuous solution of the recurrence relationship for which $\{\Gamma(x+n)\}/\{(n-1)n^x\}\to 1$ for all x. These conditions added to the interpolation problem will serve to produce a unique solution and one which coincides with Euler's. But they appear too heavy and too much like Monday morning quarterbacking. That is to say, the added conditions are hardly "natural" for they are tied in with the deeper analytical properties of the gamma function. The search went on.

Aesthetic conditions were not to be found in the older, analytic considerations, but in a newer, inner, organic approach to function theory which was developing at the turn of the century. Backed up by Cantor's set theory and an emerging theory of topology, the new function theory looked not so much at equations and identities as at the fundamental geometrical properties. The desired condition was found in notions of convexity. A curve is convex if the following is true of it: take any two points on the curve and join them by a straight line; then the portion of the curve between the points lies below the line. A convex curve does not wiggle; it cannot look like a camel's back. At the turn of the century, convexity was in the mathematical air. It was found to be intrinsic to many diverse phenomena. Over the period of a generation, it was sought out, it was generalized, it was abstracted, it was investigated for its own sake, it was applied. Called to attention by the work of H. Brunn in 1887

and of H. Minkowski in 1903 on convex bodies and given an independent interest in 1906 by the work of J. L. W. V. Jensen, the idea of convexity spread and established itself in mean value theory, in potential theory, in topology, and most recently in game theory and linear programming. At the turn of the century then, an application of convexity to the gamma function would have been natural and in order.

The individual curves which make up the gamma function are all convex. A glance at Figure 2 shows this to be true. If, as in the previous paragraph, a pseudogamma function satisfying the recurrence formula were produced by introducing the ripple $1+\sin 2\pi x$ as a factor, it would no longer be true. It must have occurred to many mathematicians to find out whether the gamma function is the only function which yields the factorial values, satisfies the recurrence relation, and is convex downward for $x>0$. Unfortunately, this is not true. Figure 5 shows a pseudogamma function which possesses just these properties. It remained until 1922 to discover a correct formulation. But it was not at too far a distance. The gamma function is not only convex, it is also logarithmically convex. That is to say, the graph of $\log \Gamma(x)$ is also convex down for $x>0$. This fact is implicit in formula (27). Logarithmic convexity is a stronger condition than ordinary convexity for logarithmic convexity implies, but is not implied by, ordinary convexity. Now Harald Bohr and J. Mollerup were able to show the surprising fact that the gamma function is the only function which satisfies the recurrence relationship and is logarithmically convex. The original proof was simplified several years later by Emil Artin, now professor at Princeton University, and the theorem together with Artin's method of proof now constitute the Bohr-Mollerup-Artin theorem. Its precise wording is this:

The Euler gamma function is the only function defined for $x>0$ which is positive, is 1 at $x=1$, satisfies the functional equation $x\Gamma(x)=\Gamma(x+1)$, and is logarithmically convex.

This theorem is at once so striking and so satisfying that the contemporary synod of abstractionists who write mathematical canon under the pen name of N. Bourbaki has adopted it as the starting point for its exposition of the gamma function. The proof: one page; the discovery: 193 years.

There is much that we know about the gamma function. Since Euler's day more than 400 major papers relating to it have been written. But a few things remain that we do not know and that we would like to know. Perhaps the hardest of the unsolved problems deal with questions of rationality and transcendentality. Consider, for instance, the number $\gamma=.57721\cdots$ which appears in formula (30). This is the Euler-Mascheroni constant. Many different expressions can be given for it. Thus,

$$\gamma = -\,d\Gamma(x)/dx\,\big|_{x=1}, \tag{36}$$

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n. \tag{37}$$

Though the numerical value of γ is known to hundreds of decimal places, it is not known at the time of writing whether γ is or is not a rational number. Another problem of this sort deals with the values of the gamma function itself. Though, curiously enough, the product $\Gamma(1/4)/\sqrt[4]{\pi}$ can be proved to be transcendental, it is not known whether $\Gamma(1/4)$ is even rational.

George Gamow, the distinguished physicist, quotes Laplace as saying that when the known areas of a subject expand, so also do its frontiers. Laplace evidently had in mind the picture of a circle expanding in an infinite plane. Gamow disputes this for physics and has in mind the picture of a circle expanding on a spherical surface. As the circle expands, its boundary first expands, but later contracts. This writer agrees with Gamow as far as mathematics is concerned. Yet the record is this: each generation has found something of interest to say about the gamma function. Perhaps the next generation will also.

The writer wishes to thank Professor C. Truesdell for his helpful comments and criticism and Dr. H. E. Salzer for a number of valuable references.

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LINEAR DIFFERENTIAL OR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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1. Introduction.* Solutions of a system of linear differential or difference equations with *real* constant coefficients a_{ij}, such as

$$(1) \qquad dx_i/dt = \sum_{j=1}^{n} a_{ij}x_j \quad \text{and} \quad x_i(t+h) = \sum_{j=1}^{n} a_{ij}x_j(t),$$

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where h is a positive constant and $i=1, \cdots, n$, are frequently desired. In discussing such systems it is usually convenient to rewrite them in the vector notation

$$\text{(2)} \qquad d\overline{X}(t)/dt = A\overline{X}(t) \quad \text{and} \quad \overline{X}(t+h) = A\overline{X}(t),$$

where the vector

$$\overline{X}(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$

and the element in the ith row and jth column of the n by n square constant matrix A is the coefficient a_{ij}. Here a bar over a capital letter is used to emphasize that a vector is under consideration.

A classical procedure for solving (2) is to first consider the corresponding matrix equations

$$\text{(3)} \qquad dX/dt = AX \quad \text{and} \quad X(t+h) = AX(t),$$

where $X=X(t)$ is an n by n square matrix $[x_{ij}(t)]$ and then to make an appropriate nonsingular linear transformation

$$\text{(4)} \qquad X(t) = PY(t),$$

which will reduce the equations of (3) to new equations of the form

$$\text{(5)} \qquad dY/dt = BY \quad \text{and} \quad Y(t+h) = BY(t),$$

where $B=P^{-1}AP$ and B has the classical Jordan canonical form.

Once the constant matrices P and B have been computed, an explicit formula for a fundamental solution $Y=Y_f(t)$ can be written out in full detail. In terms of this fundamental solution the general solutions of the respective systems (2) can be written in the form

$$\text{(6)} \qquad \overline{X} = PY_f(t) + \overline{K}_1 \quad \text{and} \quad \overline{X}(t) = PY_f(t) + \overline{K}_2(t),$$

where $\overline{K}_1$ is an arbitrary n-component constant vector and $\overline{K}_2(t)$ is an n-component vector, each component of which is an arbitrary function of t of period h.

In general, matrix P will contain elements which are complex constants. This means transformation (4), leading to the Jordan canonical form, can not in general be carried out in the real domain and this makes it awkward to deduce properties of the real solutions of the systems (1). Furthermore, in applications it is often desirable to have answer (6) in a form free from complex numbers. It is therefore the primary objective of this paper to introduce a new nonsingular *real* transformation which will lead to a *real* canonical form and a solution of relatively simple structure free from possible complex numbers.

2. Proposed real canonical forms. Turnbull and Aitken [1] have shown that any *real* system (3) can be reduced by a suitable *real* nonsingular transforma-

tion (4) to a new system (5), where matrix B has a special diagonal structure

$$(7)\qquad B=\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ & & B_p & \\ & & & C_1 \\ & & & & \ddots \\ & & & & & 0 \\ 0 & & & & & C_q \end{bmatrix} = [B_1, \cdots, B_p; C_1, \cdots, C_q].$$

To describe in detail the structure of a particular square matrix B_i, $(i=1, \cdots, p)$, appearing as a block on the main diagonal in (7), let λ_i be a real root of the characteristic equation $|A-\lambda I|=0$ and let $(\lambda-\lambda_i)^{m_i}$ be an elementary divisor of the λ-matrix $[A-\lambda I]$; then corresponding to this elementary divisor there is a square matrix on the diagonal in (7) of the form

$$B_i=\begin{bmatrix} \lambda_i & 0 & \cdots & 0 \\ 1 & \ddots & & \\ 0 & \ddots & \ddots & \\ & & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_i \end{bmatrix} \qquad (i=1, \cdots, p),$$

with m_i rows and columns.

On the other hand, if $\lambda_j=\mu_j+i\nu_j$ is a characteristic root, where μ_j and ν_j are real and $\nu_j\neq 0$, then $\bar{\lambda}_j=\mu_j-i\nu_j$ is also a characteristic root and, if $(\lambda-\lambda_j)^{m_j}$ is an elementary divisor, then $(\lambda-\bar{\lambda}_j)^{m_j}$ is also an elementary divisor. Furthermore, corresponding to this pair of elementary divisors in the complex domain, there is a real square matrix on the diagonal of (7) of the form

$$(8)\qquad C_j=\begin{bmatrix} D_j & E & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & 0 \\ & & \ddots & & E \\ 0 & \cdots & & 0 & D_j \end{bmatrix} \qquad (j=1, \cdots, q),$$

with $2m_j$ rows and columns when the elements are written out in detail, where

$$D_j=\begin{bmatrix} 0 & 1 \\ -(\mu_j^2+\nu_j^2) & 2\mu_j \end{bmatrix} \quad (\nu_j\neq 0), \qquad \text{and} \qquad E=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Although (7) is a real canonical form, for purposes of computing solutions of system (1) it is very inconvenient. In view of this fact the author then decided

to see if it were not better in (8) to replace block D_j by a new matrix

$$J_j = \begin{bmatrix} \mu_j & -\nu_j \\ \nu_j & \mu_j \end{bmatrix}$$

and then proceed to compute the corresponding solutions of (1); see [2]. Shortly thereafter, it was discovered that Niemitski and Stepanov [2] in their Russian text had already proposed this new canonical form, although the corrresponding solutions given in the text are plainly in error. Furthermore, the correct solution turned out to be rather difficult to obtain and very lengthy to write out in full detail.

3. The recommended canonical form. A later reading of a paper by Coddington and Levinson [4] suggested to the author that a far more convenient real canonical form would be obtained if in (7) each matrix C_j, corresponding to a pair of conjugate elementary divisors, were replaced by the matrix

$$G_j = \begin{bmatrix} J_j & 0 & 0 & \cdots & 0 \\ L & \cdot & & & \vdots \\ 0 & \cdot & \cdot & & \vdots \\ \vdots & & \cdot & \cdot & 0 \\ \vdots & & & \cdot & \\ 0 & \cdots & 0 & L & J_j \end{bmatrix},$$

where G_j has $2m_j$ rows and columns when the elements are written out in detail and

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrices C_j and G_j are, of course, equivalent in the sense of similarity under a real nonsingular transformation.

Once the new recommended canonical form

$$B = [B_1, \cdots, B_p; G_1, \cdots, G_q] \tag{9}$$

is adopted, systems (5) can be decomposed at once into a finite number of matrix equations in general of lower order; namely

$$dW_i/dt = B_iW_i, \qquad (10)$$

$$W_i(t + h) = B_iW_i(t), \qquad (11)$$

$$(i = 1, \cdots, p),$$

and

$$dZ_j/dt = G_jZ_j, \qquad (12)$$

$$Z_j(t + h) = G_jZ_j(t), \qquad (13)$$

$$(j = 1, \cdots, q),$$

where the solution for (5) takes the form

$$Y(t) = [W_1(t), \cdots, W_p(t); Z_1(t), \cdots, Z_q(t)]. \tag{14}$$

The advantage of this decomposition is that fundamental solutions for (10)–(13) can now be written out explicitly and in fairly compact form as follows: For (10) a fundamental solution is

$$W_i = e^{\lambda_i t}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ t/1! & 1 & & \vdots \\ t^2/2! & t/1! & & \vdots \\ \vdots & & & \\ \vdots & & \ddots & 0 \\ t^{m_i-1}/(m_i-1)! & \cdots & t^2/2! \quad t/1! & 1 \end{bmatrix}. \tag{15}$$

For (11), if $\lambda_i \neq 0$, a fundamental solution takes the form

$$W_i(t) = \lambda_i^{t/h}[\lambda_i^{m_i-1}, \lambda_i^{m_i-2}, \cdots, \lambda_i, 1] \times \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \binom{t/h}{1} & 1 & & \vdots \\ \binom{t/h}{2} & \binom{t/h}{1} & & \\ \vdots & & & \\ \vdots & & \ddots & 0 \\ \binom{t/h}{m_i-1} & \cdots & \binom{t/h}{2} \quad \binom{t/h}{1} & 1 \end{bmatrix}, \tag{16}$$

where $\binom{t}{m} = t(t-1)\cdots(t-m+1)/m!$, $m \geqq 1$; $\binom{t}{0} = 1$. In the event that $\lambda_i = 0$, then

$$W_i(t) \equiv 0. \tag{17}$$

For (12) we have the fundamental solution

$$Z_j = e^{M_j t}\begin{bmatrix} M_j & 0 & \cdots & 0 \\ tM_j/1! & M_j & & \vdots \\ t^2M_j/2! & tM_j/1! & & \vdots \\ \vdots & & & \\ \vdots & & \ddots & 0 \\ t^{m_j-1}M_j/(m_j-1)! & \cdots & t^2M_j/2! \quad tM_j/1! & M_j \end{bmatrix}, \tag{18}$$

where M_j is the 2 by 2 matrix

$$M_j = \begin{bmatrix} \cos \nu_j t & -\sin \nu_j t \\ \sin \nu_j t & \cos \nu_j t \end{bmatrix}$$

and the 0's in (18) denote 2 by 2 zero matrices.

Likewise for (13) one has the fundamental solution

$$Z_j(t) = \rho_j^{t/h}[J_j^{m-1}, J_j^{m-2}, \cdots, J_j, L]Q_j(t), \tag{19}$$

where

$$Q_j(t) = \begin{bmatrix} N_j(t) & 0 & \cdots & 0 \\ \binom{t/h}{1} N_j(t) & \cdot & & \cdot \\ \binom{t/h}{2} N_j(t) & \cdot & & \cdot \\ \vdots & & & \cdot \\ \vdots & & \cdot \quad \cdot & 0 \\ \binom{t/h}{m_j - 1} N_j(t) & \cdots & \binom{t/h}{2} N_j(t) \quad \binom{t/h}{1} N_j(t) & N_j(t) \end{bmatrix},$$

$$\rho_j = (\mu_j^2 + \nu_j^2)^{1/2}, \qquad \mu_j + i\nu_j = \rho_j e^{i\theta_j}, \qquad 0 \leqq \theta_j < 2\pi, \qquad \text{and}$$

$$N_j(t) = \begin{bmatrix} \cos(\theta_j t/h) & -\sin(\theta_j t/h) \\ \sin(\theta_j t/h) & \cos(\theta_j t/h) \end{bmatrix} \qquad (j = 1, \cdots, q).$$

It is not difficult to verify by direct substitution in (10)–(13) that these solutions (15)–(19) are correct. Furthermore, by substituting these real fundamental solutions (15)–(19) respectively in (14), one obtains a real fundamental solution $Y = Y_f(t)$, where the corresponding P in (6) is also real.

Although we now have outlined how the solutions of (1) can be found in a convenient real form, the computations required in a particular case to find the real transformation matrix P and the corresponding recommended canonical form (9) might be quite formidable. Nevertheless it is worth knowing in what direction the computational effort should be focused.

4. The trajectory pattern in a special case. Without pursuing the matter further here, as an example of an application of the real canonical form (9), we present the following

THEOREM. *If the real parts of all the characteristic roots for the differential equation* (1) *are negative, then the entire x-phase space is filled with similar, nested, nonintersecting, n-dimensional ellipsoids centered on the origin such that once any*

trajectory satisfying (1) *contacts any one of the ellipsoids it immediately enters the ellipsoid never to come out again as t increases.*

Proof. [2].

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ON THE LOCATION OF THE CENTROID OF CERTAIN SOLIDS

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The first remark that led eventually to the results of this paper came in a class discussion. The members of the class were high school teachers, participants in a National Science Foundation Institute held at Stanford University in 1957–58. The first-named author was one of the participants, the second-named author the instructor, and the remark was made by another participant, Mr. R. D. Fellman.

1. Definitions and results. A *trianguloid* is a polygon connected with a circle. The center of the circle is a special vertex of the trianguloid, called its *apex*. Those sides of the trianguloid of which the apex is not an endpoint are tangent to the circle and form jointly the *base* of the trianguloid.

Let O be the apex of the trianguloid, C_A the centroid (center of gravity) of the area of the trianguloid, C_B the centroid of its base. Both mass distributions involved are understood to be uniform; the distribution centered in C_A has constant superficial density, that centered in C_B constant linear density.

THEOREM I. *The points O, C_A and C_B lie on the same straight line, C_A between O and C_B, and $OC_A = 2C_AC_B$.*

A *pyramidoid* (sorry, not yet in the dictionary) is a polyhedron connected with a sphere. The center of the sphere is a special vertex of the pyramidoid, called its *apex*. Those faces of the pyramidoid of which the apex is not a vertex are tangent to the sphere and form jointly the *base* of the pyramidoid.

Let O be the apex of the pyramidoid, C_V the centroid of its volume, C_B the centroid of its base. Both mass distributions involved are understood to be uniform.

THEOREM II. *The points O, C_V and C_B lie on the same straight line, C_V between O and C_B, and $OC_V = 3C_VC_B$.*

When the base consists of just one side, the trianguloid is simply a triangle; in this particular case the fact asserted by Theorem I is well known. When the base consists of just one face, the pyramidoid is simply a pyramid; in that still more particular case when the pyramid is a tetrahedron, the fact asserted by Theorem II is well known. A polygon circumscribed about a circle can be regarded as a trianguloid, the base of which is the perimeter of the polygon. Similarly, a polyhedron circumscribed about a sphere can be regarded as a pyramidoid, the base of which is the surface of the polyhedron (the faces of the pyramidoid not belonging to the base degenerate by coalescing in one line which joins the center of the sphere to a vertex). What Theorems I and II state regarding circumscribed polygons and polyhedra, respectively, does not seem to be well known, and perhaps it is not known at all. Even the particular fact that the centroid C_B of the perimeter of a triangle can be so simply constructed from the well-known centroid C_A of its area and the center O of its inscribed circle, as Theorem I asserts, does not seem to be known.

Other particular and limiting cases of the two theorems will be briefly mentioned in Section 4. The proof of Theorem II will be given in Section 3 after preparatory remarks in Section 2. The proof of Theorem I is so similar to that of Theorem II that it need not be given.

2. Lemmas. Theorems I and II become intuitive if we view some familiar facts in the proper light. Let us start with a definition. Let us call two material systems *equivalent* (as seen from the standpoint of the geometry of masses) if they have the same mass and the same centroid. There are important elementary cases in which a continuous mass distribution is equivalent to a discrete mass distribution in the sense just defined.

LEMMA I. *A total mass M, distributed with constant linear density along a segment of a straight line, is equivalent to the system of two material points, each with mass $M/2$, located at the two endpoints of the segment.*

LEMMA II. *A total mass M, distributed with constant superficial density over the area of a triangle, is equivalent to the system of three material points, each with mass $M/3$, located at the three vertices of the triangle.*

LEMMA III. *A total mass M, distributed with constant density over the volume of a tetrahedron, is equivalent to the system of four material points, each with mass $M/4$, located at the four vertices of the tetrahedron.*

The language of these lemmas may be new, but the facts that they express are most familiar. For instance, to prove Lemma III, it is enough to show that the centroids of both material systems considered, the continuous and the discrete, coincide. And to prove this it is enough to show that any median plane of the tetrahedron (a plane passing through an edge and the midpoint of the

opposite edge) contains both centroids.*

LEMMA IV. *Two material systems are equivalent if they can be divided into corresponding equivalent (nonoverlapping) parts.*

Let us restate this with appropriate symbols. We express by the equation $D=D_1+D_2+\cdots+D_n$ that the mass distribution D consists of the nonoverlapping parts $D_1, D_2, \cdots, D_n$. We express by the equation $D\sim D'$ that the mass distributions D and D' are equivalent to each other in the sense defined above. With this notation we restate Lemma IV as follows:

LEMMA IV′. *If* $D=D_1+\cdots+D_n$, $D'=D_1'+\cdots+D_n'$ *and* $D_1\sim D_1', \cdots, D_n\sim D_n'$ *then* $D\sim D'$.

(Still shorter: *Equivalences between nonoverlapping mass distributions can be added.*)

To prove Lemma IV, we assume a well-known fact: If the part D_1 of the system D is replaced by one particle (material point) located at the centroid of D_1 and having a mass equal to the total mass of D_1, D is changed into another distribution with the *same centroid.*† By $2n$ consecutive changes of this nature we can change both D and D', the two mass distributions satisfying the hypothesis of Lemma IV′, into the *same* system of n material points, which renders the conclusion of Lemma IV′ evident.

3. Proof of Theorem II. Dissect each face that belongs to the base of the pyramidoid into triangles and call any triangle so obtained a *component triangle.* Call a tetrahedron of which a component triangle is the base and O, the apex of the pyramidoid, is the opposite vertex, a *component tetrahedron.* The base of the pyramidoid is dissected into component triangles and, correspondingly, the volume of the pyramidoid is dissected into component tetrahedra.

The centroid of the mass that uniformly fills the volume of the pyramidoid is C_V. We redistribute the mass within each component tetrahedron as follows: we first concentrate one quarter of the total mass contained in the tetrahedron in each vertex; then we redistribute the mass of the three vertices different from O uniformly over the base (a component triangle) of which they are the vertices. By Lemmas II, III and IV we obtain so a final mass distribution equivalent to the original one, so that C_V remains the centroid. One quarter of the total mass is now concentrated in O and three quarters distributed over the surface of the base of the pyramidoid. We assert that this distribution over the surface of the base is uniform.

In fact, by construction, the distribution is uniform within each component triangle. Let b stand for the area of this triangle, v for the volume of the com-

* See, *e.g.*, G. Pólya, How to Solve It, 2nd ed., Garden City, N. Y., 1957, p. 40.

† *Cf.* The Works of Archimedes, edited by T. L. Heath, New York, 1920, p. 194, Proposition 8. The fact becomes evident if we go back to the definition of the centroid by moments.

ponent tetrahedron of which it is the base and r for the radius of the sphere of center O that is connected with the pyramidoid. By the definition of this kind of polyhedron the plane in which the triangle with area b is situated is tangent to the sphere, and so r is the altitude of the component tetrahedron drawn from O. Therefore $v=\frac{1}{3}rb$. The mass distributed over the area b is proportional to $3v/4$ and so finally proportional to the area b. This proves the asserted uniformity of the distribution.

Now we concentrate the mass uniformly distributed over the base of the pyramidoid into its centroid C_B. The new distribution so obtained consists of two material points, O and C_B; one quarter of the total mass is located at O, three quarters at C_B. Yet, by Lemma IV, this new distribution is equivalent to the previous one and C_V is still its center of gravity. This renders Theorem II evident.

4. Remarks. Our theorems have many interesting particular cases and limiting cases. We shall mention only a few such cases.

(1) *Circumscribed polyhedra.* A polyhedron is circumscribed about a sphere with center O, the centroid of the volume of the polyhedron is C_V, the centroid of its surface C_B. If two of the three points O, C_V and C_B happen to coincide, all three points must coincide.

There is a corresponding theorem dealing with circumscribed polygons. Yet both theorems become false if "circumscribed" is replaced by "inscribed."

(2) *A sector of a circle* is obviously a limiting case of a trianguloid. The arc takes over the role of the base, its centroid is C_B. Both C_B and C_A, the centroid of the area of the sector, lie, of course, on the line of symmetry of the sector, which also passes through O, the center of the circle. By Theorem I, $OC_A=\frac{2}{3}OC_B$. The reader may verify this relation for the semicircle, which is a particular sector, by using the theorems of Pappus (which can also be used to verify a more extensive particular case of Theorem I).

(3) A *sector of a sphere* (generated by the revolution of a sector of a circle about one of its diameters) is obviously a limiting case of a pyramidoid. A zone on the spherical surface (generated by the revolving arc) takes over the role of the base, its center of gravity C_B is midway between the two parallel planes that cut off the zone from the spherical surface (this follows from the well-known fact, discovered by Archimedes, that the area of the zone is proportional to its altitude). The centroid C_B, the centroid C_V of the volume of the spherical sector and the center O of the sphere lie, of course, on the axis of revolution. By Theorem II $OC_V=\frac{3}{4}OC_B$. The hemisphere is a particular case; if the radius is r, $OC_B=\frac{1}{2}r$ and therefore (as is well known) $OC_V=\frac{3}{8}r$.

(4) Another interesting limiting case of a pyramidoid is a *sector of a circular cylinder*; any figure on the surface of the cylinder may be the base, any point on the axis of the cylinder the apex. If the sector is cut off from the cylinder by

two planes (their line of intersection and the axis of the cylinder meet in the apex) the points C_B and C_V can be located by elementary integrations.

(5) The extension of Theorems I and II to higher dimensions is clearly suggested and can be easily carried through.

A GENERALIZATION OF THE TRIGONOMETRIC FUNCTIONS

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Recently I investigated a generalization of the trigonometric functions which were similar enough in form to the sine and cosine functions to suggest interesting generalizations of many of the results of classical analysis. In particular we shall consider Euler's result on $\Gamma(\frac{1}{2})$, Euler's beta integral and the geometric roles played by the trigonometric functions in Euclidean spaces. We shall show that the functions are hyperelliptic in general and we shall investigate the circumstances under which they have a real period. We shall not consider the functions from the point of view of complex variables but shall use the methods of elementary real analysis.

The functions, which I call the alpha and beta functions are defined by the differential system

$$\frac{d}{dx}\alpha_s(x) = \beta_s^{s-1}(x), \qquad \frac{d}{dx}\beta_s(x) = -\alpha_s^{s-1}(x), \tag{1}$$

$$\alpha_s(0) = 0, \qquad \beta_s(0) = 1. \tag{2}$$

The parameter s will be called the order of the functions and will be taken in this paper to be real and an integer greater or equal to unity. Picard's theorem assures us that the system (1) and (2) have unique solutions satisfying the boundary conditions and that these functions are continuous and have continuous derivatives at least in some neighborhood of the origin.

If we multiply the two equations (1) by $\alpha_s^{s-1}(x)$ and $-\beta_s^{s-1}(x)$, respectively, we get

$$\alpha_s^{s-1}(x)\cdot\frac{d}{dx}\alpha_s(x) = -\beta_s^{s-1}(x)\cdot\frac{d}{dx}\beta_s(x) = \alpha_s^{s-1}(x)\beta_s^{s-1}(x).$$

Considering the first equality we have

$$\alpha_s^{s-1}(x)\cdot\frac{d}{dx}\alpha_s(x) + \beta_s^{s-1}(x)\cdot\frac{d}{dx}\beta_s(x) = 0.$$

If we now integrate this expression we get $\alpha_s^s(x)+\beta_s^s(x)=$constant. Setting $x=0$ and using (2) we find that the integration constant is equal to unity. We have the identity

$$\alpha_s^s(x) + \beta_s^s(x) = 1. \tag{3}$$

To show that our functions are connected with hyperelliptic functions we note that if w and u are two variables connected by the integral expression $\int_0^w (1-4t^s)^{-1/s}dt = u$, then

$$w = \alpha_s(u)\cdot\beta_s(u). \tag{4}$$

Consider the integral. If $s=2$ it is integrable by trigonometric functions, as we would expect. If $s=3$ or $s=4$, the integral is elliptic, and the alpha and beta functions are elliptic functions. When the order s is greater than 4 the integral is in general hyperelliptic. The alpha and beta functions are, therefore, a particular class of hyperelliptic functions.

From the functional equation (3) and (1) we can derive the differential equation satisfied by the alpha function. Let $x=\alpha_s(y)$, so that $dx/dy=\beta_s^{s-1}(y) = (1-x^s)^{(s-1)/s}$. If we now separate variables and integrate we get

$$y = \int_0^x (1 - t^s)^{(1-s)/s}dt = \arg\, \alpha_s(x). \tag{5}$$

Similarly,

$$\int_x^1 (1 - t^s)^{(1-s)/s}dt = \arg\, \beta_s(x). \tag{6}$$

We have used the notation arg to denote the inverse function.

Before continuing we make a definition. We define the numbers π_s or $\pi(s)$ by the relation

$$\tfrac{1}{2}\pi_s = \arg\, \alpha_s(1) = \arg\, \beta_s(0) = \int_0^1 (1 - t^s)^{(1-s)/s}dt. \tag{7}$$

It will be shown later that π_s is four times the area enclosed by the curves $x^s+y^s=1$ in the first quadrant of the xy-plane. As s increases we see that this area tends more and more closely to the unit square bounded by the x- and y-axes and the lines $x=1$, $y=1$. Hence, we state that the sequence of numbers π_s increases monotonically and approaches the limit four. This can also be concluded from an examination of the above integral. We have $\pi(1)=2$, $\pi(2) = 3.1415, \cdots, \pi(3)=3.595\cdots$ and so on. $\pi(2)$ is, of course, our ordinary pi. We might venture the hypothesis that π_s is a transcendental number for s an integer larger than unity because of our knowledge of the transcendence of $\pi(2)$. From (5) and (6) we see that

$$\alpha_s(\tfrac{1}{2}\pi_s) = 1, \qquad \beta_s(\tfrac{1}{2}\pi_s) = 0. \tag{8}$$

In another treatment the alpha and beta functions might have been defined as the inversions of the integrals (5) and (6). By the derivation of (5) and (6) we

have extended the interval over which these functions are defined to the interval $0 \leqq x \leqq \frac{1}{2}\pi_s$.

From these integrals we can easily obtain several interesting results. First we have the

LEMMA. *When their order is an even integer the alpha and beta functions are odd and even functions, respectively. That is,*

$$\alpha_s(-x) = -\alpha_s(x), \qquad \beta_s(-x) = \beta_s(x), \qquad s = 2, 4, 6, \cdots. \tag{9}$$

This, incidentally, extends the range of definition of the even-order functions to the interval $-\frac{1}{2}\pi_s \leqq x \leqq \frac{1}{2}\pi_s$.

Proof. To prove that the alpha function is an odd function we merely make the observation that when s is an even integer the integrand of (5) is even and hence the integral is an odd function of the upper limit of integration. Hence $\alpha_s(x)$ is an odd function since its inverse is odd.

To show that $\beta_s(x)$ is an even function is probably most easily done by making the observation that the derivative of $\beta_s(x)$, given by (1), is an odd function when s is even. $\beta_s(x)$ is, therefore, an even function.

We now state the second

LEMMA.

$$\alpha_s(\tfrac{1}{2}\pi_s - x) = \beta_s(x) \qquad \beta_s(\tfrac{1}{2}\pi_s - x) = \alpha_s(x) \tag{10}$$

Proof. Now

$$\int_0^1 (1 - t^s)^{(1-s)/s} dt = \int_0^x (1 - t^s)^{(1-s)/s} dt + \int_x^1 (1 - t^s)^{(1-s)/s} dt$$

which we write as

$$\tfrac{1}{2}\pi_s = \arg \alpha_s(x) + \arg \beta_s(x). \tag{11}$$

To show that this is equivalent to (10) we consider the function $\phi(x) = \alpha_s(\frac{1}{2}\pi_s - x)$. If we let $x = \arg \beta_s(u)$ we get $\phi(x) = \alpha_s(\frac{1}{2}\pi_s - \arg \beta_s(u)) = u = \beta_s(x)$. This is the first of the relations (10). Since the demonstration of the second of the relations of (10) is the same (or it can be derived from the first) the lemma is proven.

For the functions of even integer order we have

$$\alpha_s(x + \pi_s) = \alpha_s(\tfrac{1}{2}\pi_s - (-\tfrac{1}{2}\pi_s - x)) = \beta_s(\tfrac{1}{2}\pi_s + x) = \alpha_s(-x) = -\alpha_s(x).$$

Writing the first and last expressions we have

$$\alpha_s(x + \pi_s) = -\alpha_s(x), \qquad s = 2, 4, 6 \cdots. \tag{12}$$

Similarly, we find that

$$\beta_s(x + \pi_s) = -\beta_s(x), \qquad s = 2, 4, 6 \cdots.$$

If we now replace x by $x + \pi_s$ in these equations we get

$$\alpha_s(x + 2\pi_s) = \alpha_s(x), \qquad \beta_s(x + 2\pi_s) = \beta_s(x), \qquad s = 2, 4, 6 \cdots. \tag{13}$$

The alpha and beta functions of even integer order are, therefore, periodic functions with the period $2\pi_s$. The other period of the even-order elliptic and hyperelliptic functions and all the periods of the odd-order functions are complex.

To indicate the geometrical properties of these functions we begin by defining the KN_s-function by the improper integral

$$KN_s(x) = \int_0^\infty u^x e^{-(1/s)u^s} du, \qquad x > -1. \tag{14}$$

This integral can easily be brought into the form of the gamma function by making the change of variable $t = (1/s)u^s$ and can be seen to converge if $x > -1$.

If we integrate by parts we get the reduction formula $KN_s(x) = (x-s+1) \cdot KN_s(x-s)$, where x must satisfy $x > s-1$ for the latter integral to converge.

The KN_s-function was introduced to simplify the statement of the following generalization of Euler's theorem.

THEOREM. *For any a and b for which the integral converges, that is, for a, $b > -1$,*

$$\int_0^{1/2\pi_s} \alpha_s^a(\phi) \cdot \beta_s^b(\phi) d\phi = \frac{KN_s(a) \cdot KN_s(b)}{KN_s(a + b + 1)}. \tag{15}$$

Proof. The proof is essentially Euler's. We write

$$KN_s(a) \cdot KN_s(b) = \int_0^\infty \int_0^\infty x^a y^b e^{-(1/s)(x^s+y^s)} dx dy,$$

where we have used different integration variables, x and y, in the two integral factors, and we have combined the two integrals into one double integral, a surface integral over the first quadrant of the xy-plane.

In this surface integral we make the transformation

$$x = R \cdot \beta_s(\phi), \qquad y = R \cdot \alpha_s(\phi), \qquad 0 \leqq \phi \leqq \tfrac{1}{2}\pi_s, \qquad R^s = x^s + y^s. \tag{16}$$

We must show that this transformation covers the entire first quadrant once and only once as the variables R and ϕ range over the intervals $0 \leqq R \leqq \infty$ and $0 \leqq \phi \leqq \frac{1}{2}\pi_s$. This may be done as follows. We note that every point of the first quadrant lies on one and only one member of the family of curves $R =$ constant if R is restricted to the interval above. The point (x_0, y_0) lies on the curve, $x_0^s + y_0^s = x^s + y^s = R_0^s$ and so has a unique R_0 associated with it.

For any fixed R, now, as ϕ ranges over $0 \leqq \phi \leqq \frac{1}{2}\pi_s$ the corresponding point (x, y) ranges from $(R, 0)$ to $(0, R)$ and since the alpha and beta functions are continuous on the interval $0 \leqq \phi \leqq \frac{1}{2}\pi_s$ and one-valued (since, for example, from (5) and (6) their inverses are monotonically increasing and decreasing functions, respectively, on the interval) we see that each point on the curve will have one

and only one value of ϕ associated with it. Conversely, if a set (R, ϕ) is given we can see by using the same procedure that only one point corresponds to it, or by using the equations (16). We calculate the Jacobian of the transformation in the usual way

$$\frac{\partial(x, y)}{\partial(R, \phi)} = \begin{vmatrix} \beta_s(\phi) & \alpha_s(\phi) \\ -R\cdot\alpha_s^{s-1}(\phi) & R\cdot\beta_s^{s-1}(\phi) \end{vmatrix} = R[\beta_s^s(\phi) + \alpha_s^s(\phi)] = R$$

and so we find that the differential of area is

$$dA = dxdy = RdRd\phi. \tag{17}$$

The double integral then becomes

$$KN_s(a)\cdot KN_s(b) = \int_0^\infty \int_0^{1/2\pi_s} [R\beta_s(\phi)]^a [R\alpha_s(\phi)]^b e^{-(1/s)R^s} RdRd\phi$$

$$= \int_0^\infty R^{a+b+1} e^{-(1/s)R^s} dR \cdot \int_0^{1/2\pi_s} \beta_s^a(\phi)\alpha_s^b(\phi)d\phi$$

and since the first integral is exactly $KN_s(a+b+1)$ we have the result stated. Because of the symmetry of a and b on the right-hand side of (15) we may interchange a and b in the integral without altering its value.

Let us consider the transformation which we used to prove the theorem, the set of equations (16) and (17). We can interpret the transformation in two ways.

I. First, we can consider x and y as the ordinary cartesian coordinates in the Euclidean plane which are related by these equations to a system of coordinates which have the parametric curves $x^s+y^s=$constant. This is the interpretation which we used to prove the theorem.

II. We can interpret (16) and (17) as polar coordinates in a two-dimensional Minkowski space, that is, one in which the Pythagorean theorem is replaced by $R^s=x^s+y^s$. If $s=2$, of course, both these interpretations coincide.

These polar coordinates may easily be generalized to a space of any number of dimensions. For N dimensions the result is

$$x_1 = R\cdot \prod_{j=1}^{N-1} \alpha_s(\phi_j),$$

$$x_k = R\cdot\beta_s(\phi_{k-1})\cdot \prod_{j=k}^{N-1} \alpha_s(\phi_j) \qquad k = 2, 3, \cdots, N,$$

$$dV = \prod_{j=1}^{N} dx_j = R^{N-1} dR \cdot \prod_{j=1}^{N-1} \alpha_s^{N-j-1}(\phi_{j-1}) d\phi_{j-1},$$

$$R^s = \sum_{j=1}^{N} x_j^s, \qquad dQ^w = \sum_{j=1}^{N} dx_j^w.$$

We have used the symbol dQ for the differential of arc length. The number $w=2$ using interpretation I and $w=s$ using interpretation II. In the equation for x_k we take the value of the product to be unity when $k=N$. When $s=2$ the equations reduce to spherical coordinates for an N-dimensional Euclidean space.

We stated before that the number π_s is four times the area in the first quadrant bounded by the curve $x^s+y^s=1$, that is,

$$4\int_0^1 (1-t^s)^{1/s}dt = \pi_s. \tag{18}$$

We may demonstrate this in several ways, the transformation to polar coordinates probably being the simplest. To integrate this without polar coordinates we make the substitution $t=\alpha_s(\phi)$, $dt=\beta_s^{s-1}(\phi)d\phi$. Then if $|u| \leqq 1$ we find that

$$\int_0^u (1-t^s)^{1/s}dt = \int_0^\phi \beta_s^s(\phi)d\phi = \tfrac{1}{2}[\phi + \alpha_s(\phi)\beta_s(\phi)].$$

The integration formula for $\beta_s^s(\phi)$ may easily be obtained by integrating by parts. If we change variables again from ϕ to u we get the desired integration

$$\int_0^u (1-t^s)^{1/s}dt = \tfrac{1}{2}[\arg \alpha_s(u) + u(1-u^s)^{1/s}], \qquad |u| \leqq 1.$$

We find that the algebraic part of the antiderivative corresponds to a triangle while the part involving the inverse alpha function corresponds to an area bounded by a curved side in a manner which is familiar to us from the quadrature of the circle. In particular, if we set $u=1$ then we get (18).

Our interest in (18) lies in the fact that using it we can generalize Euler's result on the gamma function, $\Gamma(1/2)=\pi^{1/2}$. If we integrate (18) using the gamma function, we get the following generalization of Euler's result: if $p=2^{(N+1)}$ then

$$\Gamma(1/p) = 2^{1-(1/p)} \prod_{k=0}^{N} [2^{N-k-1}\pi_2^{N-k+1}]^{2^{-k}}.$$

If we set $N=0$ we get Euler's result. For $N=1$ and $N=2$ we get

$$\Gamma(1/4) = 2^{3/4}\sqrt{\pi_4\sqrt{\tfrac{1}{2}\pi_2}},$$
$$\Gamma(1/8) = 2^{7/8}\sqrt{2\pi_8\sqrt{\pi_4\sqrt{\tfrac{1}{2}\pi_2}}},$$

and so on.

As many results of this sort as we wish may easily be obtained by integrating, using the gamma function, expressions associated with the alpha and beta functions. Various other well-known results promise to be capable of generalization in the same way.

GENERALIZATION OF A PEANO SYMBOL

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1. Superscript sign. At the turn of the last century, G. Peano* introduced the sign $\uparrow$, but unfortunately it seems to have been overlooked by the mathematical community. He discussed its convenience for the cases $a^{m+n}=a\uparrow m+n$ and $a^{m^2+n^2}=a\uparrow m^2+n^2$; however it turns out that sign is also useful for more difficult superscript cases. For example,

$$(S^X)^{S^{m-k}} = (S^X)^{S\uparrow m-k}, \qquad 2^{2^{\aleph_0}} = 2^{2\uparrow\aleph_0} = 2\uparrow 2^{\aleph_0},$$

$$\rho w^{i_1\cdots i_p i_{p+1}\cdots i_n} = \rho w \uparrow \langle i_1 \cdots i_p i_{p+1} \cdots i_n \rangle .$$

Note that the Peano symbol is useful as well in the following case: $\exp(-i\Phi) = e\uparrow -i\Phi$. Moreover, it gives us the needed notation K. Menger† requests for the nth power and first power functions, $<\uparrow n>$ and $<\uparrow 1>$, respectively.

2. Subscript sign. The above Peano idea can easily be extended to a number of difficult subscript cases by the employment of the $\downarrow$ sign. For example, in topology, $\pi_{2i(q-1)^2}(X)$ can be more economically printed as $\pi\langle\downarrow 2i(q-1)^2\rangle(X)$, and ρ_{S^2} as $\rho\downarrow S^2$, and so on.

3. Combined subscript and superscript sign. The $\uparrow$ sign§ finds its most interesting application in conjunction with the troublesome summation sign, product sign, set-theoretical signs, etc. For example, $\sum_{i=1}^{m+n-1}\phi_i(x)$ can be more conveniently printed employing the $\uparrow$ sign in one of the following two ways:

$$\sum \langle i = 1\uparrow m+n-1\rangle \phi_i(x), \qquad \sum_{i=1\uparrow m+n-1}\phi_i(x),$$

where in all cases the left side of the arrow is the subscript and the right side of the arrow is the superscript. It can also be used advantageously in printing various mapping notations, *e.g.*, $d_2^{m+1,m}=d_{2\uparrow m+1,m}$, and

$$d_s'^{p,q}\colon E_s'^{p,q} \to E_s'^{p+s,q-s+1}$$

can be written as

$$d'\langle s\uparrow p,\, q\rangle\colon E_s'\uparrow_{p,q} \to E_s'\uparrow_{p+s,q-s+1}.$$

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* G. Peano, Formulaire de Mathematique, vol. 2, Turin, 1898.

† K. Menger, Calculus, A Modern Approach, Boston, 1952.

§ The sign introduced in Math. Mag., vol. 33, 1959, p. 24, is a less ambiguous one than the arrow.

SOLUTIONS BY QUADRATURE OF RICCATI AND SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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It is a well-known fact that the general Riccati and the general homogeneous second-order linear differential equations are equivalent, in the sense that, each being transformable into the other, a general solution of one of them entails that of the other. It is equally well known that no general method of solution by quadratures of either equation has been found. In this paper I consider (unlimited) classes of these equations for which the general solutions by quadratures are possible. Both the general Riccati and the general linear equations are first brought to the unified "normal" form, with all the coefficients lumped in the "invariant" of the equation. Using then the method here developed, we obtain the general solutions by quadratures whenever the (given) invariant is found to identically satisfy any one of a well defined infinite sequence of distinct differential equations.

1. The method. The general form of the Riccati equation, x and y supposed real, is

$$y' = e(x) + f(x)y + g(x)y^2. \tag{1}$$

Transform (1) by $y = a(x) - v'/(gv)$, where $v \equiv v(x)$ is a new dependent variable and $a(x)$ an arbitrary transformation function. (1) goes over into

$$v'' - \{f + 2ag + (g'/g)\}v' + g(e + af + a^2g - a')v = 0. \tag{2}$$

Choose now $a(x)$ such that

$$g(e + af + a^2g - a') + \{f + 2ag + (g'/g)\}' = 0, \tag{3}$$

so that (2) becomes $v'' - [\{f+2ag+(g'/g)\}v]' = 0$ and thus $v' = \{f+2ag+(g'/g)\}v + C_1$,

$$v(x) = g \exp\left\{\int (f + 2ag)dx\right\} \cdot\left[C_2 + C_1 \int g^{-1} \exp\left\{-\int (f + 2ag)dx\right\} dx\right], \tag{4}$$

where C_1, C_2 are arbitrary constants.

The general solution of (1) is then

$$y(x) = -a(x) - \frac{f}{g} - \frac{g'}{g^2} - \frac{C_1}{gv(x)} \tag{5}$$

with $v(x)$ given by (4). Now $a(x)$, which we shall call the "conjugate" to $y(x)$, satisfies, from (3), the differential equation

$$a' = e_1 + f_1 a + g_1 a^2, \tag{6}$$

where $e_1 \equiv -(1/g)\{f+(g'/g)\}' - e$, $f_1 \equiv -f - 2(g'/g)$, $g_1 \equiv -g$.

If a particular integral of (6), which is all that we need, is found on inspection, or if $e_1 \equiv 0$, or if both e_1 and f_1 are proportional to g, the general solution of (1) is immediate. Failing this, it is futile to consider (6) anew as a given equation (1) and apply the same procedure to it, since this would only give $y(x)$ as the "conjugate" to $a(x)$. We get, however, an unending sequence of distinct Riccatian equations by transforming (6) by the reciprocal function $w \equiv 1/a$, to get the differential equation

$$w' = -g_1 - f_1 w - e_1 w^2, \tag{7}$$

then obtaining the equation of the conjugate a_1 to w as

$$a_1' = e_2 + f_2 a_1 + g_2 a_1^2, \tag{8}$$

where $e_2 \equiv (1/e_1)\{(e_1'/e_1) - f_1\}' + g_1$, $f_2 \equiv f_1 - 2(e_1'/e_1)$, $g_2 \equiv e_1$, and, if a_1 is still indeterminable, forming the reciprocal $w_1 \equiv 1/a_1$, obtaining its conjugate, and so on until an equation for some a_n either gives a particular integral by inspection, has an identically vanishing coefficient e_{n+1}, or has e_{n+1}, f_{n+1} proportional to g_{n+1}. By working backwards we are led to the general solution of (5). In recovering an original w_n from its known conjugate a_{n+1}, we use (5), where, besides the obvious changes of notation, we set $C_1 = 0$ since only particular integrals are needed at all except in the last step.

2. Application to the normal equation. The normal form of $z' = e + fz + gz^2$ is obtained by the transformation $2g^2 z = 2gy - fg - g'$ and is

$$y' = p + y^2 \tag{9}$$

with the invariant $p \equiv eg + (2f'g^2 + 2gg'' - 2fgg' - 3g'^2 - f^2g^2)/(4g^2)$, so that the general solution of (9) is, from (5),

$$y(x) = -a(x) - C_1/v(x) \tag{10}$$

with

$$v(x) = \exp\left\{2\int a\,dx\right\}\left[C_2 + C_1 \int \exp\left\{-2\int a\,dx\right\} dx\right]. \tag{11}$$

The general homogeneous second-order linear differential equation $Y'' + r(x)Y' + t(x)Y = 0$ is transformed by $Y = z \exp\left\{-\frac{1}{2}\int r\,dx\right\}$ into the normal form

$$z'' + pz = 0 \tag{12}$$

with the invariant $p \equiv t - \frac{1}{4}r^2 - \frac{1}{2}r'$, and again by $z' + yz = 0$ into (9), so that the general solution of (12) is

$$z(x) = v(x) \exp\left\{-\int a\,dx\right\}$$

(13)

$$= \exp\left\{\int a\,dx\right\}\left[C_2 + C_1 \int \exp\left\{-2\int a\,dx\right\} dx\right].$$

For the unified normal equation (9), the conjugate equation (6) becomes $a' = -p - a^2$, (7) becomes $w' = 1 + pw^2$, and for (8), the equation for a_1, we have $e_2 \equiv (p'^2 - pp'' - p^3)/p^3$, $f_2 \equiv -2p'/p$, $g_2 \equiv -p$.

Leaving aside the case where a particular integral of (8) is evident, we now produce a class of differential equations solvable by quadratures, namely, those for which the invariant p satisfies identically the differential equation $e_2 = 0$; p can then have any one of the following three forms (obtained by solving $e_2 = 0$):

1. $p = -2/(x+C)^2$, where C is an arbitrary constant and we find the solutions as

$$y(x) = \frac{3B_1}{B_1 X - 3B_2 X^4} - \frac{2}{X},$$

where $X \equiv x + C$, $z(x) = B_2 X^2 - B_1/(3X)$ and B_1, B_2 are arbitrary constants. This case is trivial as it is a special case of a more general form for p for which the solution has long been known. Not so, however, for the next two cases.

2. The second form of the invariant is $p = 2c_1^3 c_2 e^{c_1 x}/(c_2 + e^{c_1 x})^2$, where c_1, c_2 are arbitrary constants. (The first form for p corresponds here to the indeterminate case for which $c_1 = 0$, $c_2 = -1$.) We find

$$a(x) = c_1 \frac{e^{2c_1 x} + 2c_1 c_2 x e^{c_1 x} - c_2^2}{(c_1 x - 2)e^{2c_1 x} - 4c_2 e^{c_1 x} - c_2^2(c_1 x + 2)};$$

this is to be substituted in (11) to give $v(x)$ after evaluating the two integrals.

3. The third form of the invariant is $p = C_1^2/\{\cos(C_1 x + C_2) - 1\}$, where C_1, C_2 are arbitrary constants. Then

$$a(x) = \frac{C_1 \sin(C_1 x + C_2) - C_1^2 x}{C_1 x \sin(C_1 x + C_2) + 2\cos(C_1 x + C_2) - 2};$$

this is to be substituted in (11).

To obtain other types of p, we now assume that $e_2 \neq 0$ and form the equation for $w_1 = 1/a_1$, obtaining its conjugate equation $a_2' = e_3 + f_3 a_2 + g_3 a_2^2$, where

$$\begin{aligned} e_3 &\equiv (1/e_2)(e_2'/e_2)' - \{(2 + 3e_2)/e_2\}p \\ &\equiv p^4(p'^2 - pp'' - p^3)^{-3}\{(p^3 + pp'' - p'^2)p^{(4)} + (2p'p'' - 6p^2p' - p\,p^{(3)})p^{(3)} \\ &\quad + (5p^4 + 7pp'^2 + 3p^2p'' - p''^2)p'' - 5p'^4 - 5p^3p'^2 + p^6\}, \end{aligned}$$

$$f_3 \equiv -2(e_2'/e_2) - 2(p'/p)$$
$$\equiv 2\,\frac{p^3p' - 3pp'p'' + p^2p^{(3)} + 2p'^3}{p(p'^2 - pp'' - p^3)},$$
$$g_3 \equiv e_2 \equiv (p'^2 - pp'' - p^3)/p^3.$$

It is seen that $e_3=0$ is distinct from $e_2=0$, although any p satisfying the latter equation will also satisfy the former, and hence new forms for p are defined by $e_3=0$ that will lead to solutions by quadratures. Proceeding in this manner we can obtain an infinite sequence of differential equations $e_n=0$ defining distinct forms for p; the successive members e_n are obtained according to the following recurrence relations:

$$e_2 \equiv -(1/p)(p'/p)' - 1, \qquad f_2 \equiv -2(p'/p), \qquad g_2 \equiv -p,$$

and for $n>1$,

$$e_{n+1} \equiv (1/e_n)\{(e_n'/e_n) - f_n\}' + g_n, \qquad f_{n+1} \equiv f_n - 2(e_n'/e_n), \qquad g_{n+1} \equiv e_n.$$

These latter are the coefficients in the conjugate equation $a_n' = e_{n+1} + f_{n+1}a_n + g_{n+1}a_n^2$.

If now a given p is found to satisfy $e_{n+1}=0$, where $n>1$, then we obtain the solution by evaluating

$$a_n = -\frac{\exp\left\{\int f_{n+1}dx\right\}}{\int g_{n+1}\exp\left\{\int f_{n+1}dx\right\}dx},$$

which, using the recurrence relations, is readily found to be

$$a_n = -\frac{\prod_{m=2}^{n} e_m^{-2}}{p^2\int p^{-2}e_n\prod_{m=2}^{n} e_m^{-2}dx}, \tag{14}$$

and then computing successively $a_{n-1}, a_{n-2}, \cdots, a_1$ from the recurrence relation $a_{n-1} = e_n^2/(e_n' - e_nf_n - e_n^2a_n)$ and $a(x)$ from $a(x) = -p^2/(p' + p^3a_1)$. Finally, this is to be substituted in (11) to give the general solutions (10) or (13). The process involves merely successive differentiations of p and the evaluation of three integrals, one in (14) and two in (11).

We observe that the general solution of $e_2=0$ is known and is itself a particular integral (containing two arbitrary constants) of the fourth-order equation $e_3=0$; this fact might enable the integration of the latter to be effected and thus the determination of a more general form for p. Similarly, the general solution of $e_n=0$ is a particular one of the equation $e_{n+1}=0$, of order higher by two.

CORRECTION

The recent paper *A look at mathematical competitions* (this MONTHLY, vol. 66, 1959, pp. 201–212) erred in implying that the Hungarian Eötvös prize competition ceased to exist in 1928. Except for a scattering of years, it has in fact been given annually. A paper by Professor Aczel, University of Debrecen, giving additional information on this interesting and successful contest, will appear in a forthcoming issue of the MONTHLY.

MATHEMATICAL NOTES

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DIFFERENTIATION OF INFINITE SERIES OF FUNCTIONS

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1. Introduction. Theorem 1 below, which appears in advanced calculus texts, has undergone many generalizations. This paper contains a generalization which is, perhaps, obtained somewhat more readily than most others. The functions under discussion are real-valued with domain the real numbers or, if specified, a closed interval of real numbers.

THEOREM 1. *Let* $\{f_n(x)\}$ *be a sequence of functions satisfying the following conditions*: (1) *Each* $f_n(x)$ *has a continuous derivative on the interval* $[a, b]$, (2) $\sum f_n(x)$ *converges to* $f(x)$ *at each point in the interval* $[a, b]$, *and* (3) $\sum f_n'(x)$ *converges uniformly to* $g(x)$ *on the interval* $[a, b]$. *Then* $f'(x)=g(x)$ *for each point in the interval* $[a, b]$.

2. Uniform convergence at a point. Let $\{f_n(x)\}$ be a sequence of functions. We say that $f_n(x)$ converges to $f(x)$ uniformly at x_0 if for each positive number ϵ there is a positive number δ and a positive integer N such that $|f_n(x)-f(x)|<\epsilon$ whenever $|x-x_0|<\delta$ and $n \geqq N$.

THEOREM 2. *If* $\{f_n(x)\}$ *is a sequence of continuous functions which converges to* $f(x)$ *then for each interval* $[c, d]$ *there is a point* x_0 *such that* $c<x_0<d$ *and* $\{f_n(x)\}$ *converges to* $f(x)$ *uniformly at* x_0.

Proof. [1], pp. 108–109.

THEOREM 3. *If the hypothesis of Theorem* 2 *is satisfied, then* $f(x)$ *is continuous at* x_0.

Proof. [1], p. 108.

3. The main theorem.

THEOREM 4. *Let $\{f_n(x)\}$ be a sequence of functions satisfying the following conditions*: (1) *Each $f_n(x)$ has a continuous derivative on the interval $[a, b]$*, (2) *$\{f_n(x)\}$ converges to $f(x)$ at each point in the interval $[a, b]$, and* (3) *$\{f_n'(x)\}$ converges to $g(x)$ uniformly at x_0. Then $f'(x_0)$ exists and is equal to $g(x_0)$.*

Proof. By the mean value theorem we have

$$f_n(x) = f_n(x_0) + (x - x_0)f_n'(x_0 + \theta_n(x - x_0)),$$

for some θ_n such that $0 < \theta_n < 1$. Now, by the uniform convergence of the sequence involved, if ϵ is a positive number there is a positive number δ and an integer N such that

$$\begin{aligned}&\left|\frac{f_n(x) - f(x)}{x - x_0} + \frac{f(x_0) - f_n(x_0)}{x - x_0} + \frac{f(x) - f(x_0)}{x - x_0} - g(x_0 + \theta_n(x - x_0))\right| \\ &= \left|\frac{f_n(x) - f_n(x_0)}{x - x_0} - g(x_0 + \theta_n(x - x_0))\right| \\ &= \left|f_n'(x_0 + \theta_n(x - x_0)) - g(x_0 + \theta_n(x - x_0))\right| < \frac{\epsilon}{3},\end{aligned}$$

whenever $0 < |x - x_0| < \delta$ and $n \geqq N$.

In addition, by the convergence of the sequences involved, there are positive integers $N_1(x)$ and N_2 such that

$$\left|\frac{f_n(x) - f(x)}{x - x_0}\right| < \frac{\epsilon}{3} \quad \text{whenever} \quad n \geqq N_1(x),$$

$$\left|\frac{f(x_0) - f_n(x_0)}{x - x_0}\right| < \frac{\epsilon}{3} \quad \text{whenever} \quad n \geqq N_2.$$

Thus, all three inequalities hold whenever $n \geqq$ maximum $(N, N_1(x), N_2)$ and $0 < |x - x_0| < \delta$. Hence,

$$\begin{aligned}\left|\frac{f(x) - f(x_0)}{x - x_0} - g(x_0 + \theta_n(x - x_0))\right| &\leqq \left|\frac{f_n(x) - f(x)}{x - x_0}\right| + \left|\frac{f(x_0) - f_n(x_0)}{x - x_0}\right| \\ &+ \left|\frac{f_n(x) - f(x)}{x - x_0} + \frac{f(x_0) - f_n(x_0)}{x - x_0} + \frac{f(x) - f(x_0)}{x - x_0} - g(x_0 + \theta_n(x - x_0))\right| \qquad (1) \\ &< 3 \cdot \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Finally,

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} g(x_0 + \theta_n(x - x_0)) = g(x_0).$$

The last equality holds by virtue of the continuity of $g(x)$ at x_0 and the bounded-

ness of $\{\theta_n\}$. The fact that, in (1), θ_n depends on x is of no consequence since it is sufficient that (1) holds for each x in $(x_0-\delta, x_0+\delta)$ and some θ_n and that the set $\{\theta_n\}$ is uniformly bounded.

We see that Theorem 4 is a generalization of Theorem 1 by noting that, if a sequence converges uniformly on an interval $[a, b]$, then it converges uniformly at each point in the interval.

THEOREM 5. *Let $\{f_n(x)\}$ be a sequence of functions satisfying the following conditions*: (1) *Each $f_n(x)$ has a continuous derivative on the interval $[a, b]$*, (2) *$\sum f_n(x)$ converges to $f(x)$ at each point in the interval $[a, b]$, and* (3) *$\sum f_n'(x)$ converges to $g(x)$ at each point in the interval $[a, b]$. Then, if $[c, d]$ is any subinterval of $[a, b]$, there is a point x_0 in $[c, d]$ such that $f'(x_0)$ exists and is equal to $g(x_0)$.*

Proof. This theorem follows from Theorems 2 and 4.

It can be shown that the hypothesis of Theorem 5 implies that the points of $[a, b]$ for which the derivative of $f(x)$ does not exist and equal $g(x)$ form a set of the first category. This, of course, is a stronger result than Theorem 5.

Reference

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ON THE DISCONJUGACY OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Let I be an interval on the x-axis with endpoints a and b $(a<b)$. I is not assumed to be closed and we denote the closure of I by I'. Let $r(x)$ and $p(x)$ be real-valued functions which are continuous over I', and assume that $r(x)>0$ there. We consider the self-adjoint second-order differential equation

$$(r(x)y')' + p(x)y = 0. \tag{1}$$

This equation is said to be disconjugate on I if and only if no nontrivial solution of (1) vanishes more than once on I. This terminology we introduced by Wintner [4]. In this note we shall show that, by applying a modified Prüfer transformation to a solution of (1), we can obtain certain sufficient conditions for (1) to be disconjugate on I. Such a transformation was introduced by Barrett [1] to study the oscillation properties of (1) over an infinite interval. It was also used by Moore in [3], although in a slightly different form from that used by Barrett.

Let $w(x)$ be a positive, continuously differentiable function over I'. Let $y(x)$ be a nontrivial solution of (1) over I'. Since $y(x)$ and $y'(x)$ cannot vanish for the same value of x the function $\rho(x)$ defined by $\rho = rw^{-1}(w^2r^{-2}y^2+y'^2)^{1/2}$ is positive, and continuously differentiable, over I'. Thus a function $\theta(x)$ is determined by

$$y(x) = \rho(x)\sin\theta(x), \qquad y'(x) = w(x)r(x)^{-1}\rho(x)\cos\theta(x), \tag{2}$$

where we may assume that $-\pi/2<\theta(a)\leqq\pi/2$. $\theta(x)$ is continuously differentiable over I'. This is true even when $y'(x)=0$: in fact, for such an x, $\cos\theta(x)=0$ and $\sin\theta(x)=1$, and so $\theta'(x)=w(x)\rho(x)/p(x)y(x)$. Using (1) and (2) we find that

$$\theta'=\frac{1}{2}\left(\frac{w}{r}+\frac{p}{w}\right)+\frac{1}{2}\left(\frac{w}{r}-\frac{p}{w}\right)\cos 2\theta+\frac{w'}{2w}\sin 2\theta. \tag{3}$$

This equation was obtained by Barrett in [1]. Our principal result is the following one.

THEOREM 1. *The equation* (1) *is disconjugate on I if there is a function $w(x)$ which is positive and continuously differentiable over I' and for which*

$$\left|\frac{w}{r}+\frac{p}{w}\right|+\left|\frac{w}{r}-\frac{p}{w}\right|+\left|\frac{w'}{w}\right|\leqq\frac{2\pi}{b-a} \tag{4}$$

for all x in I, with strict inequality for some x in I if b belongs to I.

Proof. Suppose the equation (1) is not disconjugate on I, and let $y(x)$ be a solution of (1) for which $y(a)=0$ and $y'(a)>0$. By the definition of disconjugacy there is a solution $y_1(x)$ of (1) which has at least two zeros, x_1 and x_2 $(x_1<x_2)$, in I. If $y_1(x)$ is a multiple of $y(x)$, then $y(x)$ has zeros at a and x_2. If $y_1(x)$ and $y(x)$ are linearly independent, then by the Sturm separation theorem ([2], p. 119), $y(x)$ has a zero between x_1 and x_2. In either case then, $y(x)$ vanishes at some point of I other than a. Let c be the first point to the right of a for which $y(c)=0$. Then we must have $y'(c)<0$.

From (2) we see that $y(x)$ vanishes when and only when $\theta(x)$ is an integral multiple of π. Hence $\theta(a)=0$. Also, $\theta(c)=k\pi$, and k must be odd since $y'(c)<0$. Thus $\theta(x)=\pm\pi$ for some x in I other than a. However, this is impossible if, in the (t, x)-plane, the graph of $t=\theta(x)$ stays within the triangle with vertices $(a, 0)$ and $(b, \pm\pi)$ for all x in I. In fact, if b does not belong to I, this graph may coincide with a side of this triangle. This will be the case as long as $|\theta'(x)|\leqq\pi/(b-a)$ for all x in I, with strict inequality for some x in I if b belongs to I. The condition stated in the theorem then follows from (3).

If we assume that $p(x)>0$ and that $p(x)$ and $r(x)$ are differentiable over I' and then set $w(x)=(p(x)r(x))^{1/2}$ we see that (1) is disconjugate on I if

$$\left(\frac{p}{r}\right)^{1/2}+\frac{1}{4}\left|\frac{r'}{r}+\frac{p'}{p}\right|\leqq\frac{\pi}{b-a}$$

for all x in I, with the usual condition if b belongs to I. If we again assume $p(x)>0$ over I', then there is a positive constant d such that $p(x)r(x)\geqq d^2$ for all x in I, and if we choose $w(x)=d$ we see that (1) is disconjugate on I if $p(x)\leqq\pi d/(b-a)$ for all x in I, with the usual condition if b belongs to I.

Stronger results can be obtained if we replace the sides of the triangle in the proof of Theorem 1 by the graphs of certain functions. The following general result is proved in exactly the same way as Theorem 1.

THEOREM 2. *Let $f(x)$ be a monotone, differentiable function over the interval I' such that $f(a)=0$ and $f(b)=\pm\pi$. Then the equation* (1) *is disconjugate on I if there is a function $w(x)$ which is positive and continuously differentiable over I' and for which*

$$\left|\frac{w}{r}+\frac{p}{w}\right|+\left|\frac{w}{r}-\frac{p}{w}\right|+\left|\frac{w'}{w}\right|\leqq 2\left|f'\right|$$

for all x in I, with strict inequality for some x in I if b belongs to I.

If we let $f(x)$ be either of the functions of x found by solving for t in $t^2=\pi^2(x-a)/(b-a)$, we obtain, by means of Theorem 2, the following criterion: (1) is disconjugate on I if there is a positive, continuously differentiable function $w(x)$ over I' such that

$$\left|\frac{w}{r}+\frac{p}{w}\right|+\left|\frac{w}{r}-\frac{p}{w}\right|+\left|\frac{w'}{w}\right|\leqq\frac{\pi}{(b-a)^{1/2}(x-a)^{1/2}} \tag{5}$$

for all x in I, with strict inequality for some x in I if b belongs to I. We shall now show that this criterion is actually stronger than the criterion of Theorem 1. Let I be the interval $0<x<\pi$ and consider the equation $y''+p(x)y=0$ there, where $p(x)$ is continuous over I'. If for any x_0 in I we have $p(x_0)>1$, then (4) is not satisfied at x_0 by any positive differentiable function w. To show this, note that (4) may be written $w^2+p+\left|w^2-p\right|+\left|w'\right|\leqq 2w$. If, at x_0, $p\leqq w^2$, this becomes $2w^2+\left|w'\right|\leqq 2w$, which cannot hold since $w>1$ at that point, and if $p>w^2$, this becomes $2p+\left|w'\right|\leqq 2w$, which cannot hold since $p>w^2$ and $p>1$ imply $p>w$. Thus, if the equation is disconjugate on I, this fact cannot be determined from Theorem 1.

If we set $w=1/2$, (5) becomes $1+4p+\left|1-4p\right|\leqq 2(\pi/x)^{1/2}$. Take $c>0$ and $0<d<\pi/2$ and define p by setting $p(x)=-(3+4c)x/4d+1+c$ for $0\leqq x\leqq d$ and $p(x)=1/4$ for $d\leqq x\leqq\pi$. Then for $d\leqq x\leqq\pi$, $1+4p+\left|1-4p\right|=2\leqq 2(\pi/d)^{1/2}$. For $0\leqq x\leqq d$, $1+4p+\left|1-4p\right|=8p$. Now $8p(d)=2<2(\pi/d)^{1/2}$, and the slope of the graph of $2(\pi/x)^{1/2}$ at d is $-(\pi/d^3)^{1/2}$ and the lefthand slope of the graph of $8p$ at d is $-2(3+4c)/d$. Therefore, since $2d^{1/2}(3+4c)<\pi^{1/2}$ for c and d small enough, and since the graph of $2(\pi/x)^{1/2}$, for $0<x\leqq d$, lies above its tangent line at $x=d$, we have $8p<2(\pi/x)^{1/2}$ for $0<x\leqq d$. Thus, for c and d small enough the equation in question is disconjugate on I. However, for sufficiently small x, $p(x)>1$, so this disconjugacy is not a consequence of Theorem 1.

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A NOTE ON EXTREME VALUES OF A FUNCTION OF SEVERAL VARIABLES

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In this note we consider the problem of discriminating between extreme values and saddle points of a function of n variables, in so far as this can be done from a knowledge of the first and second derivatives of the functions involved. The case where the variables are all independent is considered in the first part. We give a proof of the result which is a variant of a standard proof. In the second part the method is extended to the case where the n variables are connected by $m(<n)$ equations, and Lagrange's undetermined multipliers are used to determine stationary values. A precise result is obtained in this case as well.

Let x denote the vector $(x_1, \cdots, x_n)$, with real components, $f(x)$ a real function of x. Set $f_r(x)=\partial f(x)/\partial x_r$ and $f_{rs}(x)=\partial^2 f(x)/\partial x_r \partial x_s$. The norm of x, namely $\sqrt{(x_1^2+\cdots+x_n^2)}$ will be denoted by $|x|$. $f(x)$ has an extreme value at $x=z$ if there exists a positive real number ϵ such that $0<|y|\leqq\epsilon$ implies that $f(z+y)-f(z)$ is of like sign. If $f_r(x)=0$ for all r, and if for every ϵ, $f(z+y)-f(z)$ takes negative as well as positive values on the surface of the sphere $|y|=\epsilon$, then z is a saddle point of $f(x)$.

Our first result is this:

If $f(x)$, $f_r(x)$ are all differentiable functions at $x=z$, and $f_r(z)=0$ for all r, then $f(x)$ has an extreme value at $x=z$ if the matrix $\|f_{rs}(x)\|$ is definite; $f(x)$ has a saddle point at $x=z$ if the matrix is neither definite nor semidefinite.

We quote the following standard result, the proof of which will be omitted.

If $\|a_{rs}\|$ is a symmetric matrix, and if M and m are its greatest and least characteristic roots, then the maximum and minimum values of $y'(a_{rs})y$ on the surface of the sphere $|y|=\epsilon$ are $M\epsilon^2$ and $m\epsilon^2$. It follows that if M and m are of the same sign, the matrix is definite but it is semidefinite if either is zero.

Now set $\phi(t)=f(z+ty)$, where t is a real scalar; then

$$\phi(1)-\phi(0)=f(z+y)-f(z) \tag{1}$$

and we have

$$\phi'(t)=\sum_{r=1}^{n} y_r f_r(z+ty). \tag{2}$$

Since $f_r(x)$ is differentiable at $x=z$,

$$f_r(z+ty)=f_r(z)+t\sum_{s=1}^{n} y_s f_{rs}(z)+to(|y|).$$

Thus since $f_r(z)=0$ for all r, we have from (2): $\phi'(t)=ty'(a_{rs})y+to(|y|^2)$, where $a_{rs}=f_{rs}(z)$. Hence $\phi(1)-\phi(0)=\int_0^1\phi'(t)dt=\frac{1}{2}y'(a_{rs})y+o(|y|^2)$. The result now follows from (1) and the lemma.

We now consider the case where the variables $x_1, \cdots, x_n$ are not independent but are connected by $m(<n)$ equations: $\psi^k(x)=0$; $k=1, \cdots, m$ is a suffix and not an exponent. $\psi^k(x)$ and $\psi^k_r(x)$ are differentiable functions of x for each k and r. Using Lagrange's undetermined multipliers $\alpha_1, \cdots, \alpha_m$, set $F(x) = f(x) - \sum_{k=1}^{m} \alpha_k \psi^k(x)$ and form the n equations $F_r(z)=0$. Together with the m equations $\psi^k(z)=0$ these determine z and the values of the α_k.

Set $\phi(t)=F(z+ty)$ and $A_{rs}=F_{rs}(z)$. As before, we get

$$\phi(1) - \phi(0) = \tfrac{1}{2}y'(A_{rs})y + o(|y|^2). \tag{3}$$

Since $\psi^k(z+y)=\psi^k(z)=0$ for the values of y under consideration, we have $\phi(1)-\phi(0)=f(z+y)-f(z)$. Hence, from (3) we get

$$f(z+y) - f(z) = \tfrac{1}{2}y'(A_{rs})y + o(|y|^2). \tag{4}$$

Now $f(x)$ certainly has an extreme value at $x=z$ if the matrix $\|A_{rs}\|$ is definite. It may, however, have an extreme value even if $\|A_{rs}\|$ is not definite or semidefinite. The reason is that the vectors y in (4) are not completely arbitrary. If $v^k=(\psi^k_1(z), \cdots, \psi^k_n(z))$ and $y \to 0$ in such a way that $z+y$ always satisfies the conditions $\psi^k(z+y)=0$, then the scalar product $(y, v^k)=o(|y|)$. Thus in (4), y can be taken to be orthogonal to each of the v^k. If $u_1, \cdots, u_{n-m}$ are $n-m$ independent vectors, each orthogonal to all the v_k, then any other vector orthogonal to every v_k is a linear combination of the vectors $u_1, \cdots, u_{n-m}$. If U is the matrix whose columns are $u_1, \cdots, u_{n-m}$ and $w=(w_1, \cdots, w_{n-m})$ is an arbitrary vector with $n-m$ components, then any vector y orthogonal to every vector v_k is of the form $y=Uw$. Hence from (4) it is sufficient if $w'U'(A_{rs})Uw$ is of the same sign. Thus for $f(x)$ to have an extreme value it is sufficient for the square symmetric matrix $U'(A_{rs})U$ of $n-m$ rows and columns to be definite. Further, if this matrix is not definite or semidefinite then there is no extreme value at the point z.

THE INEQUALITY OF STEENSHOLT FOR AN n-DIMENSIONAL SIMPLEX

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Let P be an arbitrary point inside the S_n simplex of n-dimensional space. Let A_i $(i=1, \cdots, n+1)$ be the vertices of the simplex, and $S^{(i)}_{n-1}$ the $(n-1)$-dimensional boundary simplex opposite to A_i $(i=1, \cdots, n+1)$. Let us denote the distance of the point P from the vertex A_i by $\overline{PA}_i=R_i$, and the distance of the point P from the boundary simplex $S^{(i)}_{n-1}$ by $\overline{PS}^{(i)}_{n-1}=r_i$. Furthermore, let C_n be the n-dimensional content of S_n, and $C^{(i)}_{n-1}$ be the $(n-1)$-dimensional content of $S^{(i)}_{n-1}$ $(i=1, \cdots, n+1)$. Then the following inequality holds:

$$\sum_{i=1}^{n+1} C^{(i)}_{n-1} R_i \geqq n \sum_{i=1}^{n+1} C^{(i)}_{n-1} r_i. \tag{1}$$

Proof. If $A_iS_{n-1}^{(i)}=h_i$ $(i=1, \cdots, n+1)$ are the altitudes of S_n, then $C_n = C_{n-1}^{(i)}h_i/n$ $(i=1, \cdots, n+1)$. Summing up these equalities we get $(n+1)C_n = (1/n)\sum_{i=1}^{n+1} C_{n-1}^{(i)}h_i$. But because of the inequality $h_i \leqq R_i+r_i$ $(i=1, \cdots, n+1)$, it follows that

$$n(n+1)C_n \leqq \sum_{i=1}^{n+1} C_{n-1}^{(i)}(R_i + r_i),$$

$$= \sum_{i=1}^{n+1} C_{n-1}^{(i)}R_i + \sum_{i=1}^{n+1} C_{n-1}^{(i)}r_i.$$

But $nC_n = \sum_{i=1}^{n+1} C_{n-1}^{(i)}r_i$; thus we get the required result (1) by a rearrangement. The inequality (1) for triangles was proved by Steensholt [1]. Later its validity was extended by Thébault [2] for tetrahedrons.

If we introduce the weighted values R and r, which were applied by Steensholt, we obtain

$$R = \left(\sum_{i=1}^{n+1} C_{n-1}^{(i)}R_i\right) \Big/ \left(\sum_{i=1}^{n+1} C_{n-1}^{(i)}\right),$$

$$r = \left(\sum_{i=1}^{n+1} C_{n-1}^{(i)}r_i\right) \Big/ \left(\sum_{i=1}^{n+1} C_{n-1}^{(i)}\right)$$

and hence $R \geqq nr$.

If it is the case that $C_{n-1}^{(i)} = C_{n-1}^{(k)}$ $(i, k=1, \cdots, n+1)$, then

$$\sum_{i=1}^{n+1} R_i \geqq n \sum_{i=1}^{n+1} r_i \tag{2}$$

and this is always true in the case of regular simplices. Relation (2) holds, however, in the case of such simplices, that have congruent boundary simplices (so that their contents are equal), but are not regular (an isosceles simplex). Such simplices exist in those spaces in which it is possible to inscribe a regular simplex in the cube. These simplices, however, are not orthocentrical, and so in this case the equality does not hold. In the case of tetrahedrons having equal boundary simplices (tetrahedron isosceles) inequality (2) is known [3]. Thus (2) is the extension of the Erdös-Mordell inequality [1, 2] for the case of an n-dimensional regular simplex and an isosceles simplex.

The equality holds only when P is the orthocenter of the regular simplex.

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NOTES ON DIFFERENTIAL GEOMETRY

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We define an ovaloid as a convex closed surface with continuous nonvanishing principal curvatures ($K>0$). Such a surface has two parallel tangent planes in every plane direction (opposite tangent planes). When the distance between opposite planes is the same for all directions, we call the ovaloid a surface of constant width.

A surface of constant width has the following properties:

(i) the principal directions at opposite points are parallel;

(ii) the sums of the principal radii of curvature that correspond to parallel principal directions at any pair of opposite points is constant;

(iii) all its normals are double [1];

(iv) the mean curvatures at opposite points are equal [2].

Conversely, any surface which has property (iii) or (i) and (ii) is of constant width [1].

On the basis of the above, we shall discuss further properties of the surface and prove the following theorems.

THEOREM 1. *Let K and K' be the Gaussian curvatures of the surface of constant width μ at any pair of opposite points P, P' on the orthogonal section of any cylinder circumscribed about the surface, and K_n and K_n', the normal curvatures at P, P' in the direction of the orthogonal section. Then*

$$\frac{\pi K_n}{K}+\frac{\pi K_n'}{K'}$$

is constant and equal to the perimeter of the orthogonal section.

Proof. Let the lines of curvature be taken as the parametric curves $v=$constant on the surface of constant width, the corresponding principal radii of curvature and principal curvatures being ρ_a, ρ_b and K_a, K_b, respectively. Let us assume that if $\mathbf{N}$ is the unit normal vector at a point, then the unit normal vector $\mathbf{N}'$ will be chosen in such a way that $\mathbf{N}=-\mathbf{N}'$. Furthermore, let θ be the angle between the direction $v=$constant and the generators of the cylinder at the point P and ψ the angle between the direction $v=$constant and the direction of the orthogonal section at the same point. Then the radius of curvature R of the orthogonal section is [1]

$$R=\rho_a\cos^2\theta+\rho_b\sin^2\theta,$$

that is,

$$\begin{aligned}KR&=K_b\cos^2\theta+K_a\sin^2\theta\\&=K_b\sin^2\psi+K_a\cos^2\psi\\&=K_n\qquad\text{(by Euler's theorem).}\end{aligned}$$

Since the principal directions at P, P' are parallel, the angle will be the same at P, P'. Hence we have also $K'R'=K_n'$. But [1], $R+R'=\mu$ and therefore we obtain $(K_n/K)+(K_n'/K')=\mu$. Hence

$$\frac{\pi K_n}{K}+\frac{\pi K_n'}{K'}=\pi\mu. \tag{1}$$

Now, the number on the right of (1) is equal to the perimeter of the orthogonal section of the circumscribed cylinder and it is constant for any such cylinders about the surface of constant width μ. Hence our theorem is proved.

THEOREM 2. *If the magnitude of the normal component of the sum of the vector curvatures of the lines of curvatures through any point P of a surface of constant width μ is denoted by n_P and that at the opposite point P', by $n_{P'}$, then*

$$n_P = n_{P'}, \tag{2}$$

each being equal to twice the mean curvature of the surface at P or P'.

Proof. Let the surface have the parametrization as before and let the vector curvatures of $v=$constant and $u=$constant be denoted by $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. We may decompose $\mathbf{k}_i$ $(i=1, 2)$ into two components, $\mathbf{k}_{in}$ normal to, and $\mathbf{k}_{ig}$ tangential to, the surface. Then the sum of the vector curvatures of the lines of curvature may be written

$$\begin{aligned}\mathbf{k}_1+\mathbf{k}_2 &= \mathbf{k}_{1n}+\mathbf{k}_{1g}+\mathbf{k}_{2n}+\mathbf{k}_{2g}=\mathbf{k}_{1n}+\mathbf{k}_{2n}+\mathbf{k}_{1g}+\mathbf{k}_{2g}\\ &= K_a\mathbf{n}+K_b\mathbf{n}+\mathbf{k}_{1g}+\mathbf{k}_{2g}.\end{aligned}$$

Its normal component is $K_a\mathbf{n}+K_b\mathbf{n}$ and hence we obtain

$$(2J=)\ K_a+K_b=n_P.$$

Similarly, we have at P',

$$(2J'=)\ K_a'+K_b'=n_{P'}.$$

Thus, by virtue of (iv), we obtain (2) and our theorem is proved.

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CLASSROOM NOTES

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CONCERNING DOMAINS OF REAL FUNCTIONS

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By the domain of a function ϕ, I mean the set of all x for which $\phi(x)$ exists.

In modern textbooks of calculus the concept of domain is emphasized when real functions are defined, but is apt to take a back seat thereafter. For example, in Begle's [1] a theorem

$$\{\phi + \psi\}' = \{\phi' + \psi'\}$$

is stated (the prime denotes differentiation). This is not quite true: the most that is proved is $\{\phi+\psi\}' \supseteq \{\phi'+\psi'\}$. The subject is evidently a tricky one.

The usual instances of functions with restricted domains are too trivial to be effective. If ϕ is defined by

$$\phi(x) = 2x + 5 \quad \text{whenever} \quad 0 \leqq x \leqq 2,$$

so having for domain the closed interval $[0, 2]$, the restriction appears artificial. If a student is told that the domain of the function ψ defined by

$$\psi(x) = x^{-1} \text{ whenever the right-hand side exists}$$

is the set of all *nonzero* numbers, he merely thinks "Of *course* it is."

It is therefore interesting that a paradox in maxima-and-minima depends on the domain of a function not being properly determined. The paradox is Example 2 on page 517 of Thomas [3]: To find the least value of $x^2+y^2+z^2$ when $x^2-z^2-1=0$.

We argue that the least value will be a local minimum; so we eliminate one variable, say z, from the minimand via the given equation, and look for local minima of the result; *i.e.*, of $2x^2+y^2-1$. Its partial derivatives are $4x$ and $2y$; these are simultaneously zero only when $x=0$ and $y=0$, . . . and then z is undefined. Geometrical intuition suggests that there *is* a minimum, so we try again, this time eliminating x instead of z, and find minima at $(1, 0, 0)$ and $(-1, 0, 0)$. But this does not save the paradox; there is no logical reason why the second try should necessarily avoid the snag.

We must define "local minimum" carefully. There are two reasonably common definitions:

1. ϕ has a local minimum at a if there is an open interval N containing a such that $\phi(a) \leqq \phi(x)$ whenever x is in N.

2. ϕ has a local minimum at a if there is an open interval N containing a such that $\phi(a) \leqq \phi(x)$ whenever x is in N and *also in the domain of* ϕ.

The first definition is equivalent to that in Courant [2]; the second to that in Begle [1].

The difference shows most often at the end of an interval: if

$$\alpha(x) = +\sqrt{(1 - x^2)} \text{ with domain } [-1, 1],$$

then α has local minima at 1 and -1 under Definition 2 but not under Definition 1.

It is now obvious how to define a local minimum of a function under a given condition. *E.g.*:

1*. ϕ has a *local minimum* at (a, b) under the condition $\mathfrak{P}$ if $\mathfrak{P}(a, b)$ is true and there is an open interval N containing (a, b) such that $\phi(a, b) \leqq \phi(x, y)$ whenever (i) (x, y) is in N and (ii) $\mathfrak{P}(x, y)$ is true.

2*. is the same but for the addition of "and (iii) (x, y) is in the domain of ϕ."

It is also clear how to define a minimum under a condition:

ϕ has a *minimum* at (a, b) under $\mathfrak{P}$ if $\mathfrak{P}(a, b)$ is true and $\phi(a, b) \leqq \phi(x, y)$ whenever (i) (x, y) is in the domain of ϕ, and (ii) $\mathfrak{P}(x, y)$ is true.

The highbrow method of finding minima when the condition is expressible by equations is by Lagrange's multipliers. I am interested in lowbrow methods; I do not mean nonrigorous, but simple and obvious. The usual lowbrow method is to eliminate variables, so getting a function of fewer variables, and to find the minima of this. In our example, this function was χ, where

$$\chi(x, y) = 2x^2 + y^2 - 1. \tag{1}$$

Let us consider the general case: the minima of $\phi(x_1, \cdots, x_{m+n})$ under n conditions $\psi_i(x_1, \cdots, x_{m+n}) = 0$. We can eliminate the last n variables if the equations are equivalent to $x_i = \xi_i(\mathbf{x})$ for i from $m+1$ to $m+n$ where $\mathbf{x}$ is short for $x_1, \cdots, x_m$. Let us just consider this case; there is not much loss of generality. If the function got by elimination is χ, then

$$\chi(\mathbf{x}) = \phi(\mathbf{x}, \xi_{m+1}(\mathbf{x}), \cdots, \xi_{m+n}(\mathbf{x}))$$

and its domain is the set of all $\mathbf{x}$ for which $(\mathbf{x}, \xi_{m+1}(\mathbf{x}), \cdots, \xi_{m+n}(\mathbf{x}))$ is in the domain of ϕ. With this definition of χ it is easy to prove that ϕ has a minimum at $(a_1, \cdots, a_{m+n})$ under the given conditions if and only if χ has a minimum at $\mathbf{a}$ and $a_i = \xi_i(\mathbf{a})$ for i from $m+1$ to $m+n$.

In our example, we now see that formula (1) for χ is incomplete. It needs the legend "for every x and y for which there is a z such that $x^2 - z^2 - 1 = 0$ and $x^2 + y^2 + z^2$ is defined." This is clearly equivalent to "for every x and y for which $|x| \geqq 1$." Minima are to be suspected (i) where the partial derivatives are all zero, (ii) where they do not all exist, and (iii) on the boundary of the domain of χ. Here (i) and (ii) yield nothing; (iii) yields $x = 1$ or -1, $z = 0$, y arbitrary. Investigating our suspects, we find that $y = 0$ does give minima.

Thus, although minima of χ correspond to those of ϕ under $\mathfrak{P}$ if Definition 1 is used, local minima do not necessarily correspond to local minima. In fact,

whether a minimum of ϕ under $\mathfrak{P}$ corresponds to a local minimum of the function we get by elimination may depend on which variables we eliminate.

The lowbrow method works if students are taught to look for all three types of suspected extrema. Either definition of local extremum, properly presented, will lead them to do so, *provided* that the domain of χ is not neglected.

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SOME SIMPLE EXAMPLES OF GROUPS*

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Generally speaking, it would seem advisable to leave "track-covering" to Rogers' Rangers when they are being pursued by hostile Senecas, and to lay out, for our students to follow with ease, the paths by which we attain our mathematical results. Occasionally, however, in order to manufacture examples and exercises, we may be justified in keeping some professional secrets, at least for a time. For instance, in first presenting to students the concept of an abstract group, we may find it useful to have at hand a supply of systems with exotic operations, without immediately divulging their source.

Examples. Such systems are easily constructed by establishing isomorphisms with groups consisting of certain subsets of the complex numbers, in which the operation is either addition or multiplication. A few examples will make clear the process:

I. Let $\{C_0; \cdot\}$ be the multiplicative group of all complex numbers except 0. Set up the correspondence $x \leftrightarrow x'+1$. We thus obtain the Abelian group $\{C_{-1}; *\}$, in which C_{-1} is the set of all complex numbers except -1, and $*$ is defined by $u * v = uv+u+v$.

II. Let $\{C; +\}$ be the additive group of all complex numbers. Set up the correspondence $x \leftrightarrow k/x'$, for any complex $k \neq 0$. We thus obtain the Abelian group $\{C_0'; *\}$, in which C_0' consists of the set of all complex numbers except 0, augmented by another element which we denote by "∞," and $*$ is defined by

$$u * v = \frac{uv}{u+v}, \quad u, v \text{ both different from } \infty;$$

$$u * \infty = u.$$

III. Applying the correspondence $x \leftrightarrow (x')^3$ to $\{R; +\}$, the additive group of reals, we obtain the group $\{R; *\}$, where $u * v = \sqrt[3]{(u^3+v^3)}$.

IV. Application of the correspondence $x \leftrightarrow (x'+1)^2$ to $\{C_0; \cdot\}$ leads again

* The author gratefully acknowledges stimulating conversations with R. G. Long, C. O. Oakley, and E. C. Schlesinger.

to the group of Example I.

V. If we apply the correspondence $x \leftrightarrow x'+1$ to the group $\{1, 2, 3, 4; \times\}$ in which the operation is multiplication, mod 5, we are led to the group $\{0, 1, 2, 3; *\}$, in which $u * v$ represents $uv+u+v$, mod 5.

It is instructive for the student to search for subgroups of these "artificial groups." In doing so, he may remind himself that $\{0; \cdot\}$ constitutes a group.

Generalizations. A detailed analysis of the "linear isomorphisms" of Examples I and II seems worthwhile.

Consider the correspondence

$$x \leftrightarrow \frac{ax' + b}{cx' + d}, \tag{1}$$

where a, b, c, d are complex numbers, $ad-bc \neq 0$, $a \neq 0$, $c \neq 0$.

I'A. If (1) is applied to $\{C_0; \cdot\}$ we obtain the Abelian group $\{S; *\}$, where S consists of the symbol ∞, together with all complex numbers except $-b/a$ and $-d/c$, and $*$ is defined as follows:

(i) For $u \neq \infty, v \neq \infty$, $u * v = (Luv+M(u+v)+N)/(Puv+Q(u+v)+R)$, if the denominator is not equal to zero; $u * v = \infty$ otherwise.

(ii) For $u \neq \infty$, $u * \infty = (Lu+M)/(Pu+Q)$, if $Pu+Q \neq 0$; $u * \infty = \infty$ otherwise.

(iii) Finally, $\infty * \infty = L/P$, if $P \neq 0$; $\infty * \infty = \infty$ otherwise.

In the foregoing equations, $L=a^2d-bc^2$; $M=bd(a-c)$; $N=bd(b-d)$; $P=ac(c-a)$; $Q=ac(d-b)$; $R=ad^2-b^2c$.

I'B. If $c=0$ in the correspondence (1), we are led to the Abelian group $\{S'; *\}$, where S' consists of the set of all complex numbers except $-b/a$, and $*$ is defined by

$$u * v = (Luv + M(u + v) + N)/R.$$

If $a=0$, $c \neq 0$, we obtain nothing new.

II'A. If (1) is applied to $\{C; +\}$ we obtain the Abelian group $\{S''; *\}$, where S'' consists of the symbol ∞, together with all complex numbers except $-d/c$, and $*$ is defined as in I'A, with the values of the constants as follows: $L=bc^2-2acd$; $M=-ad^2$; $N=-bd^2$; $P=ac^2$; $Q=bc^2$; $R=2bcd-ad^2$. No special attention is required if $a=0$.

II'B. If $c=0$ in II'A, we are led to the Abelian group $\{C; *\}$, where $u * v = (M(u+v)+N)/R$.

It is clear that the complex numbers could be replaced by the reals, or by the rationals, throughout the discussion of I' or II' without altering the validity of the results. An attempt to reverse the processes of I' and II' leads to interesting consequences.

Suppose that an operation $*$ is defined by

$$u * v = \frac{Luv + M_1u + M_2v + N}{Puv + Q_1u + Q_2v + R}.$$

It can be verified that, if $*$ is to be associative and nontrivial, M_1 must equal M_2 and Q_1 must equal Q_2, *i.e.*, the operation must be commutative. Kuwagaki showed [1] that

$$u * v = \frac{Luv + M(u + v) + N}{Puv + Q(u + v) + R} \tag{1}$$

is associative if and only if

$$\text{rank of} \begin{bmatrix} M & L - Q & P \\ N & M - R & Q \end{bmatrix} = 1. \tag{2}$$

If condition (2) is satisfied, the operation defined by (i), modified as in (ii) and (iii) to accommodate the symbol ∞, permits an identity element and an inverse to such element.

The group to be defined depends upon the nature of the simultaneous zeros of the numerator and denominator of (i). In the extended complex plane, the conics

$$Lxy + M(x + y) + N = 0, \qquad Pxy + Q(x + y) + R = 0,$$

may have

(α) Two distinct finite points of intersection, (p, q) and (q, p);
(β) A doubly-counting finite point of intersection, (r, r);
(γ) Two doubly-counting ideal points of intersection; or
(δ) The entire ideal line in common.

In case (α), the system $\{T; *\}$ is a group if T is the set consisting of ∞ and all complex numbers except p and q, and if $*$ is defined by (i), (ii), and (iii). It can be verified that there are just two linear correspondences which effect isomorphisms of this group with $\{C_0; \cdot\}$—these are given by $x \leftrightarrow k(x'-p)/(x'-q)$ and $x \leftrightarrow (x'-q)/\{k(x'-p)\}$ for a particular k which can be computed. Example IV illustrates a "nonlinear" isomorphism.

In case (β), the system $\{T; *\}$ is a group if T is the set consisting of ∞ and all complex numbers except r, and if $*$ is defined as before. There are indefinitely many linear correspondences which effect isomorphisms of this group with $\{C; +\}$; these are given by $x \leftrightarrow \{k(Mx'+N)\}/(x'-r)$, unless $M=0=N$, in which case $x \leftrightarrow \{k(Px'+L)\}/x'$. In both instances k is arbitrary.

The situation for cases (γ) and (δ) are similar to those for (α) and (β), respectively. The values of a, b, c, d in the correspondence are most easily expressed in terms of the constants in (i); they will not be given explicitly here.

Some curious results appear if one examines subgroups in these systems. For

example, let $\{T; *\}$ be the group consisting of the symbol ∞ and all complex numbers except i and $-i$, the operation $*$ being given, for the elements other than ∞, by

$$u * v = \frac{uv - 1}{u + v}.$$

In accordance with the foregoing analysis, this group is isomorphic to $\{C_0; \cdot\}$, a correspondence being given by

$$x \leftrightarrow \frac{x' - i}{x' + i} \quad \text{or} \quad x' \leftrightarrow -\frac{i(x+1)}{x-1}.$$

The same correspondence establishes an isomorphism between $\{R_0; \cdot\}$, the multiplicative group of all reals except 0, and $\{T'; *\}$, where T' consists of ∞ and all the real multiples of i except $\pm i$. But the system $\{R'; *\}$, in which R' consists of ∞ and all the reals, is a group which is not isomorphic to any subgroup of $\{R_0; \cdot\}$ or of $\{R; +\}$.

Reference

1. A. Kuwagaki, Sur l'equation fonctionelle $f(x+y) = R\{f(x), f(y)\}$, Mem. Coll. Sci. Univ. Kyoto. Ser. A., vol. 26, 1950, pp. 139–144.

A "STATIC" APPROACH TO DERIVATIVES

IAIN T. ADAMSON, The Queen's University, Belfast

This note is a sequel to my earlier article, *A new approach to limits* [1], but can be read quite independently of it and is intended to be much less formal.

One of the most fundamental situations we have to consider in analysis is the following: $f(x)$ is a function of the real variable x, a is a real number and there is a real number L such that

(1) *for all real numbers x sufficiently close to a, $f(x)$ is arbitrarily close to L.*

It is pleasant to image a competition in which entrants would be invited to compose a short sentence which could be used as a summary description of the situation (1); unfortunately the first prize has already been awarded to the sentence

(2) *"the limit of $f(x)$ as x tends to a is L."*

Unless we share Professor Menger's doubts about the local status of x (see his article [2]) we cannot quarrel with this award on logical grounds; but there is room for some discussion of the psychological and pedagogical effects of this standard terminology. There can be no doubt that generations of schoolmasters and university lecturers have been influenced by it to describe the limit situation in kinematic or organic terms; informal discussions in books and classrooms are packed with such phrases as "x approaches a," "$f(x)$ gets closer to L,"

"h becomes smaller," and so on—the very name "limiting process" has an organic ring to it.

As a result of these discussions the sentence (2) must often conjure up in students' minds a picture rather like the following. A red bead, called x, moves along the x-axis towards a, while a white bead slides along the curve $y=f(x)$ always remaining vertically above the red one; as the red bead approaches a—which by some mysterious self-denying ordinance it never reaches—the white bead approaches the point (a, L). It may well be that such moving pictures enable students to compute limits correctly; but it is extremely doubtful whether they can ever lead to any proper appreciation of the limit situation (1), which involves no reference to motion, variation, growth or diminution—and very properly not, if we accept Frege's contention that variation can only occur in time, while analysis has nothing to do with time. To sum up, one is tempted to say that the familiar intuitive description of limits to which we are led by the conventional terminology fails to give any understanding of the fundamental situation.

Professor Menger has proposed replacing (2) by "the limit of f at a is L," and in my earlier article I have suggested that the "fundamental limit situation" which we ought to consider is rather more general than (1); I shall recall the details a little later. The reasons which led to these proposals were unconnected with the preceding discussion—roughly speaking, Menger wanted to eliminate x, while I wanted every function to have a limit at every point. But it is unlikely that either of us will win acceptance for our suggestions unless we can provide an informal and intuitive description of them which will take the place of the beloved moving beads.

What I offer here is not an intutive description of this general limit itself, but rather a new approach to the tangent problem, which provides a very natural motivation for the introduction of the limit notion. So let f be a function, a a real number; our problem is to find a definition for the tangent to the graph of f at the point $(a, f(a))$ on it. A mathematically unsophisticated student, who is sure that he knows perfectly well what the tangent is, will find it hard to stomach such a bald statement of the problem; but his acceptance can be gained —and our mathematical consciences salved—if we tell him that what we are looking for is a precise and workable definition which will reflect formally his intuitive notion of the tangent; certainly it is unsatisfactory to say that our problem is "to find the gradient of the tangent."

We now proceed as follows. Let h be any positive real number, and for every nonzero real number k between $-h$ and h consider the chord joining $(a, f(a))$ to $(a+k, f(a+k))$. This chord has gradient

$$Cf_a(k) = \frac{f(a+k) - f(a)}{k}.$$

If we examine some simple cases, where the graph of the function has an "obvi-

ous" tangent, there is little difficulty in convincing even the most reluctant student that in these cases this "obvious" tangent—the line which he feels intuitively ought to be called the tangent—lies somewhere between the steepest and flattest of all these chords. He can also be persuaded to agree that this is so for every positive real number h, no matter how small.

Thus for these simple cases we should be inclined to say that the gradient of the "obvious" tangent lies between the maximum and minimum of the numbers $Cf_a(k)$ corresponding to nonzero numbers k between $-h$ and h. But this statement needs a slight refinement: for the set of numbers $Cf_a(k)$ may not actually have a maximum or minimum—for example, if $f=I^3$ (the cubing function) and $a=0$, then $Cf_0(k)=k^2$ and the set of numbers k^2 corresponding to the *nonzero* numbers k between $-h$ and h has no minimum. The set of numbers in question will, however, have a greatest lower bound m_h and a least upper bound M_h, which are the formal substitutes for the informal notions of minimum and maximum. We can say, then, that the gradient of the "obvious" tangent lies between m_h and M_h for every positive real number h.

This now suggests the following general definition. If f is any function, a a real number, m_h and M_h the greatest lower and least upper bounds of the numbers $Cf_a(k)$ for all nonzero k between $-h$ and h, and if there is exactly one real number L which lies between m_h and M_h for all positive real numbers h, then we define the tangent to the graph of f at $(a, f(a))$ to be the line through that point with gradient L; if there is more than one number lying between m_h and M_h for all h, then the graph does not have a tangent at $(a, f(a))$.

Since we may regard the numbers $Cf_a(k)$ as the values of a function Cf_a, "the chord function of f at a," we now see that our informal discussion makes the approach to limits which I described in [1] appear a very natural one. In a simplified form, which will suffice for our present purpose, the method is as follows. If ϕ is any function, b a point of accumulation of the domain of ϕ, then the *limit set* of ϕ at b, $\lim_b \phi$, is the collection of real numbers common to all the sets $\mathrm{Cl}(E_h)$ for every positive real number h. Here E_h denotes the set of values assumed by ϕ for all real numbers between $b-h$ and $b+h$ except b itself, and $\mathrm{Cl}(E_h)$ is its closure, *i.e.*, the set obtained by adjoining to E_h all its points of accumulation. If the limit set consists of a single real number L, then we say that ϕ is *convergent* at b to L. I have shown in [1] that in this case ϕ has the limit L at b in the classical sense.

This method is perfectly workable in practice and specific results can in many cases be computed directly: the limit of $\sin/I$ at 0($\lim_{x\to 0} (\sin x/x)$ in the classical notation), for instance, can be obtained by slight modifications of the familiar "approach" procedure. It is possible also to derive directly from the definition the usual rules for the limits of the sum and product of two functions which are convergent at a point b, so that all the familiar limit results can be obtained.

Now in general the closure of E_h need not be the whole interval $[m_h, M_h]$. But if ϕ is convergent to L at b, it is not difficult to see that L is the only real

number common to all the intervals $[m_h, M_h]$; the converse is, of course, trivial. Thus we can sum up by saying that if the chord function of f at a, Cf_a, converges at 0 to L, then we define the tangent to the graph of f at $(a, f(a))$ to be the line through that point with gradient L.

There is nothing new about the result; all that is new is the method by which we have reached it. I claim three advantages for this method: first, it is completely "static"; second, the informal and intuitive descriptions with which we can introduce it actually run parallel to the precise, formal procedure, and are not, as in the classical approach, a misleading kinematic picture of an essentially static situation; and third—this from experience—it is eminently teachable.

References

1. Iain T. Adamson, A new approach to limits, Math. Gaz., vol. 42, 1958, p. 87.
2. Karl Menger, What are x and y? Math. Gaz., vol. 40, 1956, p. 256.

A USEFUL INTEGRAL FORMULA

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Integrals of the type $\int e^{ax} \sin bx dx$ are bothersome because the formulas are confusing to remember and direct integration is tedious to perform. The formula developed here is relatively easy to remember and can be used in a variety of situations.

Consider $\int f(x)g(x)dx$ where $f''(x) = hf(x)$ and $g''(x) = kg(x)$. Thus f (and of course g) is circular or hyperbolic sine or cosine, or else exponential. Note that $k\int g(x)dx = g'(x)$. Integration by parts gives

$$k\int f(x)g(x)dx = f(x)g'(x) - \int f'(x)g'(x)dx.$$

Again integrating by parts, we have

$$k\int f(x)g(x)dx = f(x)g'(x) - f'(x)g(x) + h\int f(x)g(x)dx.$$

Solving algebraically for the integral we have the formula

$$\int f(x)g(x)dx = \frac{1}{h-k}\left[f'(x)g(x) - f(x)g'(x)\right] + c \qquad h \neq k.$$

Examples are:

$$\int e^{ax} \sin bx dx = \frac{1}{a^2+b^2}\left[ae^{ax} \sin bx - be^{ax} \cos bx\right] + C.$$

$$\int \cosh ax \sin bx dx = \frac{1}{a^2+b^2}\left[a \sinh ax \sin bx - b \cosh ax \cos bx\right] + C.$$

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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THE EDUCATION OF MATHEMATICS TEACHERS*

Geometry

B. E. MESERVE, Montclair State College

Any comments upon the geometry which teachers should study must be based upon some assumptions regarding the point of view and the contents of the geometry that they will teach. I shall assume that in the near future there will be a greater use of algebra in geometry, especially in the earlier introduction and use of coordinates, that a first course in geometry will treat both plane and solid geometry, that methods of proof and methods of solving problems will gradually gain greater recognition than formal proofs of specified theorems.

The traditional training of people to teach geometry seems to me to be as follows: The first course in college mathematics has a varied content but usually includes coordinate plane geometry, in some cases coordinate geometry of three space. If the first course is calculus, the analytic geometry is given only an incidental or practical treatment relative to the calculus. The traditional and, from many points of view, misnamed course of college geometry extends skills in constructions and relationships among traditional geometric figures. When there is a methods course this usually includes a scanty review and discussion of plane geometry. A course in the foundations of geometry can be very helpful and is becoming more common.

Excluding the last-named course, I suggest that we need a definite reorientation of our point of view and probably our course content if we are to prepare geometry teachers to do more than perpetuate the routines to which they have been subjected. At present it seems to me that we do reasonably well in developing skills for solving traditional problems. We must also develop both appreciation and skill regarding geometry as a deductive system, the algebraic solutions of geometric problems as in coordinate geometry, the place of secondary school geometry in modern geometry, in modern mathematics, and in our present society.

As an example of the place of coordinate geometry and its utility, consider this theorem about two chords in the same or equal circles: if we have unequal chords, the larger chord is nearer the center. In the traditional synthetic approach the proof of this theorem often involves a theorem about two triangles

* Edited transcriptions of talks by Meserve, Walter, and Fender, given as part of a panel discussion at the annual meeting of the New Jersey Section of the Mathematical Association of America on November 1, 1958. Transcriptions of the panel talks by Meder, Lorch, and Tucker appeared in this MONTHLY, vol. 66, pp. 805–809.

having two sides of one equal to two sides of the other and the relative sizes of the third side, assuming a certain order relation of the included angles. I believe that this is the only common application of the theorem on varying an angle of a triangle. From the point of view of coordinate geometry, the theorem about chords in the circle can be reduced very simply to the statement that if unequal quantities are subtracted from equal quantities, the differences are unequal in the opposite sense. If you think about it, you will see that it falls out very readily at a level of activity that can be done very nicely in the tenth grade.

The training of teachers should include a discussion of geometric figures as they are recognized in elementary schools, as they are introduced or described in junior high school, and as sets of points, lines and planes. This training should include Cartesian coordinate systems on a number line and on a plane. We should teach the extension of synthetic and coordinate geometry to three space and the coordinate geometry of higher dimensions. A comparison of the structure of high school geometry sequences and of the proofs of theorems in several of these sequences seems desirable. Common and advanced problems should be solved using methods of synthetic geometry, coordinate geometry, locus theorems and, at least in the fifth-year program, vector spaces.

I shall on several instances make reference to what I shall call a fifth-year program. We should not feel that the education of any teacher is completed at the end of four years. There are several topics that might well be introduced in the undergraduate program with the details considered more thoroughly in the fifth year.

Also in the background of the teacher, it seems to me, we should have the algebraic basis for classical constructions; the content and structure of secondary school mathematics; the use of visual aids in teaching geometry; geometry as a deductive system including finite postulational systems and isomorphic representations. Also included should be the hierarchy of geometries that one obtains by considering topology; then, as a special case, real projective geometry; as special cases, affine geometry and the non-Euclidean geometries; as a special case of affine geometry, the geometry of similar figures; and, again as a special case, Euclidean geometry. It seems to me that at least the characteristic properties of these geometries can be considered at the undergraduate level; the detailed development may need to be postponed until the fifth year.

Another topic that I have found useful and desirable can be considered from either of two points of view: the development of Euclidean geometry from projective geometry or the gradual generalization of Euclidean geometry to obtain projective geometry. In either case the treatment should include both synthetic and analytic methods. There should also be a discussion of the dependence of various theorems and properties of Euclidean geometry upon the postulates for projective geometry and the assumptions that one makes in specializing projective geometry to obtain Euclidean geometry.

Each prospective teacher should study the historical evolution of our concept of geometry. The treatment should include early empirical procedures,

the development of postulational systems, the role of artists in the development of descriptive and projective geometries, the role of algebra in the development of analytic (or coordinate) geometry, and, at least in the fifth year, the role of calculus in the development of differential geometry and the role of logic in the development of geometries as deductive systems.

Our undergraduate program should include some non-Euclidean geometry and other geometries. There should be at least a comparison of Euclidean geometry and the geometry on a sphere, recognizing that spherical geometry is not a non-Euclidean geometry but that there is an elliptic geometry of diameters of a sphere analogous to the geometry on the ideal plane. The fifth year program should include polarities as a basis for the "scratch equation" and the development of the non-Euclidean geometries from projective geometry.

Topology should be studied—at least the Jordan Curve Theorem, the four-color problem, traversable networks, the Moebius strip and, in the fifth year, a detailed study of homomorphic figures.

Finally, the program should include geometry as a study of properties of figures invariant under a group of transformations—for example, the rigid motions, translations, rotations (point and line), and, at least in the fifth year, orthogonal line reflections. The matrix representation of transformations is very effective.

It seems clear that such a breadth of background in geometry cannot be covered in a single 3 semester hour course. At the University of Illinois and also at Montclair State College about 8 semester hours are needed for reasonable thoroughness in the treatment of the background and a very brief treatment of methods. Some of the participants at the Mideast Regional State College Conference on Science and Mathematics Teacher Education* last March felt that they could not spare one fourth of the student's college mathematics program for geometry. I propose to you that the one-fourth figure is reasonable for prospective teachers of secondary school mathematics.

In conclusion, I would like to say that the discussions of the teaching of geometry at the International Congress of Mathematicians in Edinburgh this summer were the liveliest that I participated in. The problems seem to be similar in practically all countries.

Probability and Statistics

R. M. WALTER, Douglass College, Rutgers, The State University

The Commission on Mathematics of the College Board has done a great deal of the background thinking on the teacher training problem as it refers to probability theory and statistical inference, and has published its findings in the form of a proposed textbook to be used in teaching probability and statistical inference to high school students. If we are to think of the training of teachers in this field it is of interest to note briefly what the Commission thinks these teach-

* A report of the Mideast Conference may be found in the June 1958 issue of *School Science and Mathematics.*

ers should be teaching. The Commission's book contains, first, a very brief treatment of descriptive statistics. It then turns to probability theory which it introduces first in an intuitive way and then in a rather more formal way. In introducing it, it makes use of some of the algebra of sets and the concept of a sample space, and I might say in this connection that if there is any place in the high school curriculum where the use of sets will fall into place easily and naturally it is here. I know of no neater way to develop such ideas as conditional events, mutually exclusive events, etc., than by use of set notions and set algebra.

The ideas of random variable and probability distribution are introduced although the only distribution which is developed and exploited to any extent is the binomial distribution. Then, under the heading of statistics and applications, the first application described is that of acceptance sampling. Some development of the notion of a test of a hypothesis is presented and, finally, in an additional chapter which was added to the book during the past summer, some study is included of the properties of means of samples from a finite population and, using the ideas here developed, a very brief introduction to principles of statistical estimations based on sample means is given.

The introduction of this subject into the high schools, which is proposed, presents a difficult problem from the point of view of the in-service teachers. For these teachers most of the needed training will have to be provided through extension courses and summer institutes. On the other hand it would help greatly if we could get across to these people that, when they go away on vacation and take along a book to read, they might well make it a mathematical book. I would like to make a couple of suggestions along that line. These will be things that high school teachers should genuinely enjoy reading.

The first suggestion I would offer would be the book by Kemeny, Snell and Thompson, *Introduction to Finite Mathematics*. In this readers will find elementary probability theory presented in a setting in which it is preceded by symbolic logic and some of the algebra of sets. They will see probability theory developed further than is the case in the high school course that I described. Simple stochastic processes, particularly Markoff processes, are also discussed. All this is done in such a way that a high school teacher should find it enjoyable and profitable reading. I would further recommend that high school teachers who think they may someday teach probability and statistical inference, read *Elements of Probability Theory* by Harold Cramer. This book goes rather thoroughly into the foundations of probability theory. It develops the notions of random variables and their distributions much more fully than anything I have mentioned heretofore, including the well-known continuous distributions which are used in statistical inference.

In considering the college program at the advanced undergraduate or major level for college undergraduates who expect to become teachers there will be some variation. That is, there will be some students who have never had any work in probability and statistics and there will be others who may have had

some introduction to it. In any event, the equivalent of, let us say, a year's work in this area, if that much time can be allowed, certainly belongs in the undergraduate curriculum at this time. I believe I would describe an ideal course using currently available material as consisting of about one semester in which they would study as much as they could of William Feller's *Introduction to Probability Theory and Its Applications* and then an additional semester using a book on statistics which would emphasize random variables and their distributions and applications. An example of the sort of book that I have in mind is the well-known text by Paul Hoel; there are others available at about the same level. Possibly not all students will be able to include as much work in this area as I have called for here. If they have to be content with one semester's training in statistics beyond what they might get at freshman level, the book by Cramer which I mentioned before contains what I would regard as the subject matter that should be presented.

Effect of Computing Machinery

F. G. FENDER, Rutgers, The State University

I have been asked to tell you what effect the increasing use of automatic computers should have on the training of future teachers. To do this, I must give you some idea of what a computer does, and how it is made to do it. A computer is simply an information converter; that is, it operates on input data to produce output data. The input data are strings of alphabetic or numeric symbols which tell the machine what to do, how to do it, and what to do it with. Internally, the symbols are converted into a common code, and only by the function which each performs can a letter be told from a number, or a machine instruction from a piece of working data. The resulting apparent confusion is tolerated because of the increased flexibility that allowed an instruction to be treated as a number, and, by using arithmetic, to be altered to form a new instruction for a new situation. In addition to the four obvious operations of arithmetic, a computer uses instructions which take in or give out information, which separate the symbol strings into smaller units or combine them into larger groupings, or which use predetermined conditions to choose alternative sequences of events. This makes of the computer much more than just a fast desk calculator, for it can complete tasks that might be called clerical, or editorial, or even executive.

The obvious use for a computer is to do long and tedious calculations. In such work, the processes of both the differential and integral calculus are reduced to their finite difference approximations. Thus a problem in partial differential equations might be reduced to a set of simultaneous linear algebraic equations, or an integral might be reduced to the parabolic rule, essentially to a process of addition.

It is less obvious, but more important for our discussion, that a computer may be used to handle problems that are linguistic in nature. The Univac was used to produce, in record time, the Concordance of the Revised Standard Version of the Bible. The IBM 650 is being used for language studies of the Dead

Sea Scrolls and of Latin manuscripts. Translations from Russian into English are on the verge of becoming successful. Similarly, the technique used in translation from the machine code of one computer to that of another is quite well known, using either computer for the translation. Exact relationships of any form may be handled, such as the rules for differentiation of the elementary transcendents, or the determination of the greatest common factor of two polynomials. It is in this general area of linguistics that I would expect to find the most vigorous development, the most difficult problems, and the greatest interest.

Many college students, upon meeting a computer for the first time in a senior course, have a startling new experience. The machine has zero intelligence quotient, and zero emotional quotient. It is incapable of guessing what the student intended, and it will merrily carry out any task, no matter how useless. Those students who have failed to develop a personal sensitivity to mathematical thinking often begin to mature with this experience. Such shock therapy would be beneficial for prospective teachers of secondary students.

What kind of person is needed to control a computer? You may be startled to have a mathematician tell you that a computer installation does not always need mathematicians. But the installation does need people who can think clearly and who can perform the mathematician's supreme function of defining a problem in a form that leads to a solution. Often this function can be supplied by a major in philosophy, or history, or a language, who has a broad outlook, and who is well disciplined in his field. Personality is important too, for the men and women who keep a computer busy depend on one another so much there is no place for the surly boor who must work alone. Generally we need the type of mind that is interested in intellectual puzzles or games and for which every day brings a fresh learning experience.

How should such a person be trained? Let me say first that he should not be untrained; that is, he should not have his curiosity dulled. Particularly in arithmetic, a teacher's enthusiasm is as important as knowledge, and the impact of decimals, of compound fractions, and so on, can be electrifying to a child. Second, he should be allowed to experience the thrill of hard work well done. This leads to an inner intellectual honesty that is often difficult to gain, and it leads to the type of discipline that is needed to run a computer or do any research in any field. Third and last, to help our teachers make learning a vivid and thrilling experience we must do our part to upgrade the difficult and often scorned profession of teaching, particularly the teaching of all levels of mathematics. We can do this, in part, by telling the public what it is we do when we do mathematics, and, in part, by helping to create the intellectual atmosphere in which teaching is a noble and rewarding profession.

Summer Fellowships for Secondary-School Teachers

The National Science Foundation has announced a program of fellowships for secondary-school teachers of science and mathematics. Under these fellowships, teachers in secondary schools may apply for one, two, or three summers' study in a college or university of their choice. The fellowships will be awarded to support individually planned programs of graduate-level study in the mathematical, physical, and biological sciences. It is not necessary that the fellow be enrolled for or complete an advanced degree, but it is necessary that his studies be at the graduate level.

Applications for fellowships should be sent to: Secondary-School Fellowships, American Association for the Advancement of Science, 1515 Massachusetts Avenue, N.W., Washington 5, D. C. The deadline for filing applications is January 15, 1960, and awards will be announced on March 15. The universities or colleges chosen by the fellows for their studies will be notified at a later date.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1391. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $m_0, m_1, \cdots, m_r$ be positive integers which are pairwise relatively prime. Show that there exist $r+1$ consecutive integers $s, s+1, \cdots, s+r$ such that m_i divides $s+i$ for $i=0, 1, \cdots, r$.

E 1392. *Proposed by R. K. Guy, University of Malaya, Singapore, Malaya*

I am confronted by stamp machines dispensing supplies of 3 cent and 4 cent stamps. I notice that if I have a sufficient supply of coins I can obtain stamps to the value of any whole number of cents except for 1 cent, 2 cents, and 5 cents. If the machine dispenses only stamps of values m cents and n cents, how many exact values in cents can I not make up?

E 1393. *Proposed by D. S. Mitrinovitch, University of Belgrade, Belgrade, Yugoslavia*

Evaluate the nth order determinant $D=|a_{ik}|$ for the two cases: (1) $a_{ik}=0$ when $i+k$ is even, (2) $a_{ik}=0$ when $i+k$ is odd.

E 1394. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a trihedral angle and a point within it, construct the plane through the point which intercepts on the trihedral angle the tetrahedron of minimum volume.

E 1395. *Proposed by Albert Wilansky, Lehigh University*

Which of the sequences whose nth terms follow are convergent? $1/(n \cos n)$, $1/(n^2 \cos n)$, $1/(n^3 \cos n)$, $\cdots$; $1/(2^n \cos n)$; $1/(n! \cos n)$.

SOLUTIONS

An Inequality for a Triangle

E 1361 [1959, 312]. *Proposed by M. S. Klamkin, AVCO Research and Development*

If A, B, C are angles of a triangle, show that

$$\csc A/2 + \csc B/2 + \csc C/2 \geqq 6.$$

I. *Solution by Leon Bankoff, Los Angeles, Calif.* Consider the angle bisectors AD, BE, CF, concurrent at the incenter I of the triangle ABC. It is known that the sum of the ratios in which a point within a triangle divides the cevians of this point is never less than 6 (E 1043, 1953, p. 421). Since the inradius $r \leqq (ID, IE, IF)$, it follows that $AI/r+BI/r+CI/r \geqq 6$.

II. *Solution by Bernard Greenspan, Drew University.* Let

$$f = \csc A/2 + \csc B/2 + \csc C/2.$$

While f has no maximum value ($A/2$ can be chosen as near to 90° as desired), a minimum clearly exists. Consider A and B as independent variables. If we set $\partial f/\partial A$ and $\partial f/\partial B$ equal to zero, we see f is least when

$$\csc A/2 \cot A/2 = \csc B/2 \cot B/2 = \csc C/2 \cot C/2. \tag{1}$$

Since $\csc \phi \cot \phi$ is a strictly decreasing function as ϕ varies from 0° to 90°, (1) and $A+B+C=180°$ imply $A=B=C=60°$. Therefore the least value of f is 3 csc 30° or 6.

III. *Solution by L. D. Goldstone, New York State Public Works Lab., Albany, N. Y.* By a well-known theorem: If the sum of n angles, each positive and less than $\pi/2$, is given, the sum of the cosecants of the angles is least when the angles are all equal (Hobson, *Treatise on Plane Trigonometry*, 4th ed., 1918, p. 88). Hence for a convex n-gon, $\sum \csc A/2 \geqq n \csc \pi(n-2)/2n$. The value $n=3$ yields the desired result.

IV. *Solution by W. J. Blundon, Memorial University of Newfoundland.* In the theorem of Toth (*Lagerungen in der Ebene auf der Kugel und im Raum*, p. 11): "If O is an arbitrary point in the plane of a triangle ABC of inradius r, then $OA+OB+OC \geqq 6r$, with equality only for an equilateral triangle," we take O to be the incenter and the result follows at once.

Note. The theorem quoted is a special case of a deeper theorem with the inequality $OA+OB+OC>2\sqrt{(T\sqrt{3})}$, where T represents the area of the triangle ABC. This latter theorem is an immediate consequence of the isoperimetric property of the hexagon $AO_3BO_1CO_2$, where O_1, O_2, O_3 are the images of O in the sides BC, CA, AB of the triangle.

V. *Solution by Viktors Linis, University of Ottawa.* Applying the Erdös-Mordell inequality (this MONTHLY, 1958, p. 521) to the center I of the inscribed circle of radius r in the triangle ABC we obtain $IA+IB+IC\geqq 6r$. Since $IA=r\csc A/2$, $IB=r\csc B/2$, $IC=r\csc C/2$, the required inequality follows, with equality only for $A=B=C$.

VI. *Solution by J. S. Frame, Michigan State University.* We use the following two lemmas:

$$u-1\geqq \ln u \text{ for } u>0, \qquad \text{equality if and only if } u=1,$$

$$\cos^2 A/2=\sin^2 (B+C)/2=\sin B\sin C+\sin^2 (B-C)/2\geqq \sin B\sin C.$$

Then

$$\begin{aligned}\csc A/2-2&\geqq 2\ln\left[(1/2)\csc A/2\right]=2\ln\left[(\cos A/2)/\sin A\right]\\&=\ln\left[(\cos^2 A/2)/\sin^2 A\right]\geqq \ln(\sin B\sin C/\sin^2 A)\\&=\ln(bc/a^2).\end{aligned}$$

Replacing A by B and C in turn we find

$$\csc A/2+\csc B/2+\csc C/2\geqq 6+\ln(bc/a^2)+\ln(ca/b^2)+\ln(ab/c^2)=6.$$

Equality holds only for the equilateral triangle.

VII. *Solution by D. S. Passman, General Electric Co., Syracuse, N. Y.* For concave functions in general

$$\sum_1^n f(x_k)\geqq n[f(\textstyle\sum x_k)]/n.$$

Thus for $f(x)=\csc x$, $f''(x)=(1+\cos^2 x)/\sin^2 x>0$, $0<x<\pi$, $n=3$, and $\sum x_k=\pi/2$, the result follows.

Also solved by A. N. Aheart, D. A. Breault, R. F. Brown and Joel Levy (jointly), Yi Chang, P. L. Chessin, Anton Glasser, Michael Goldberg, H. W. Gould, Alfred Gray, W. E. Kesler, H. C. Liu, D. C. B. Marsh, Leo Moser and E. L. Whitney (jointly), D. L. Muench, Margaret Olmsted, J. L. Pietenpol, N. R. Riesenberg, L. A. Ringenberg, D. A. Robinson, M. T. Salhab, C. M. Sandwick, Sr., J. Schopp, Jack Silver, Stoddart Smith, Jr., Charles Wexler, R. H. Wilson, Jr., and Dale Woods.

A Hula Hoop Problem

E 1362 [1959, 312]. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

Consider a vertical girl whose waist is circular, not smooth, and temporarily at rest. Around the waist rotates a hula hoop of twice its diameter. Show that,

after one revolution of the hoop, the point originally in contact with the girl has traveled a distance equal to the perimeter of a square circumscribing the girl's waist.

I. *Solution by J. D. E. Konhauser, HRB-Singer, Inc., State College, Pa.* The locus of a point on a circle of radius $2a$ rolling along the inside of a circle of radius a is a cardioid of length $8a$, which, if a is the radius of the girl's waist, is the perimeter of a square circumscribing the girl's waist.

II. *Solution by Leo Moser, University of Alberta.* Consider the hoop as fixed and the poor girl whirling around (it serves her right). The original point of contact traverses the diameter of the hoop twice, and this is the required distance.

Also solved by Leon Bankoff, H. F. Batie, H. F. Bechtell, R. E. Blewster, Jr., A. P. Boblétt, Mark Bridger, J. F. Burke, J. H. Butchart, Allan Chertok, Stuart Friedman, Michael Goldberg, L. D. Goldstone, H. W. Hickey, R. T. Hood, Richard Holt, E. L. Hubbard, Wally Manheimer, Jr., D. C. B. Marsh, Helen M. Marston, C. S. Ogilvy, A. St. V. Parker-Jervis, J. L. Pietenpol, T. A. Porsching, D. A. Robinson, C. M. Sandwick, Sr., Sister M. Stephanie, Eric Sturley, J. A. Tierney, T. C. Wales, Charles Wexler, Patty Whitlock, and the proposer.

A Digital Root

E 1363 [1959, 312]. *Proposed by Leo Moser, University of Alberta*

Let numbers be written to base b where b has the form $b=r^2+1$. Given r consecutive numbers, the last divisible by r, then the digital root of their sum is $1+2+\cdots+r=r(r+1)/2$.

Solution by N. J. Fine, Institute for Advanced Study. The digital root d of a number N expressed to the base b satisfies: (i) $d\equiv N \pmod{b-1}$, (ii) $1\leqq d \leqq b-1$. If

$$N = (kr+1) + (kr+2) + \cdots + (kr+r) = kr^2 + r(r+1)/2,$$

then, since $r^2=b-1$,

(i) $\quad r(r+1)/2 \equiv N \pmod{b-1}, \qquad$ (ii) $\quad 1 \leqq r(r+1)/2 \leqq r^2 = b-1.$

Thus $d=r(r+1)/2$.

Also solved by H. F. Bechtell, Mark Bridger, P. L. Chessin, Monte Dernham, J. W. Ellis, H. B. Emerson, H. W. Hickey, J. H. Hodges, J. Hooley, D. C. B. Marsh, J. L. Pietenpol, and the proposer.

Editorial Note. The case $b=10$ is an old result and was given by Nicomachus about 330 A.D.

An Area Property of Roulettes

E 1364 [1959, 312]. *Proposed by James Serrin, University of Minnesota*

In a plane, let A denote a closed convex curve in contact with a given curve C. Also, let B denote the mirror image of A across the tangent line to C at the point of contact. Suppose that the curvatures of A, B, C permit A and B to roll without slipping along their respective sides of C. Then, as A rolls along C, let A' denote the roulette traced out by the point of A initially in contact with

C. Similarly, let B' denote the roulette generated by rolling B on C. Show that the area enclosed by an arch of A' and the corresponding arch of B' is independent of C. In particular, if A is a circle the enclosed area is just six times the area of A.

Solution by C. S. Ogilvy, Hamilton College. Let C be traversed by the moving point $x=x(t)$. For time Δt_i, the area increment ΔS_i can be approximated by a wedge-shaped quadrilateral two of whose sides are normal to C and bisected by C and the other two of which are equal respectively to Δx_i+h and Δx_i-h and are parallel to the direction of C at x_i. Such elements form a Darboux sum, independent of the form of C:

$$S = \lim_{n\to\infty,\Delta x_i\to 0} \sum_{i=1}^{n} 2\Delta x_i f(x_i),$$

where $f(x)$ is the length of the normal at x from C to either roulette.

In particular, we know that the area under one arch of a cycloid is three times the area of the generating circle. Add another three for the image circle; and we have just shown that it is not necessary that the base curve be a straight line.

Also solved by Michael Goldberg, D. C. B. Marsh, M. S. Klamkin, and the proposer.

Goldberg used the method of E 1269 [1958, 45] and its extension by R. C. Yates (this MONTHLY, vol. 66, 1959, pp. 130–135), obtaining the desired result as a limiting case of rolling polygons.

Similar arguments to the above can be given to show that the sum of the lengths of an arch of A' and the corresponding arch of B' is independent of C.

Polynomials and Sequences

E 1365 [1959, 312]. *Proposed by Melvin Hausner, Stevens Institute of Technology*

Let H be the class of polynomials $f(x)$ with rational coefficients such that $f(n)$ is an integer when n is an integer. Prove that a sequence a_n of $+1$'s and -1's is of the form $a_n=(-1)^{f(n)}, f(x)\in H$, if and only if it is a periodic sequence of period 2^k.

Solution by N. J. Fine, Institute for Advanced Study. If f is a polynomial of degree d, then

$$f(x) = \sum_{r=0}^{d} \Delta^r f(0) \binom{x}{r}.$$

If $f(x)$ is an integer for $x=0, 1, 2, \cdots, d$, then $\Delta^r f(0)$ is also an integer, for $r=0, 1, 2, \cdots, d$. Let $2^k>d$. Then

$$\binom{n+2^k}{r} \equiv \binom{n}{r} \pmod 2$$

for all integers n and for $r=0, 1, \cdots, d$. [This is most easily seen by equating

coefficients of x^r in

$$(1+x)^{n+2^k} = (1+x)^n(1+x)^{2^k} = \sum_{r=0}^{n}\binom{n}{r}x^r(1+x^{2^k}) \pmod 2.]$$

It follows that $(-1)^{f(n)}$ has period 2^k.

Conversely, suppose that $a_n=(-1)^{g(n)}$ has period 2^k. Define the polynomial

$$f(x) = \sum_{r=0}^{2^k-1} \Delta^r g(0)\binom{x}{r}.$$

We have $f(x)\in H$, so $(-1)^{f(n)}$ has period 2^k. But $f(n)=g(n)$ for $n=0, 1, \cdots, 2^k-1$, so $a_n=(-1)^{f(n)}$ for the same values of n. Since both have period 2^k, they agree for all n.

Also solved by J. H. Hodges, D. C. B. Marsh, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4875. *Proposed by J. M. Gandhi, Jain Engineering College, Panchkoola, India*

Prove the following conjecture: if

$$\frac{x\cosh x}{\sin x} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{D_{2n}x^{2n}}{(2n!)},$$

then

$$D_{2n} = G_{2n} + \sum_{\substack{(p-1)\mid 2n \\ p\equiv 1(4)}}\frac{1}{p} + \sum_{\substack{p\le 2n+1 \\ p\equiv 3(4)}}\frac{\epsilon_p}{p},$$

where the G_{2n} are integers and the ϵ_p are integers depending on p and n. Indeed an explicit formula for ϵ_p can be obtained.

4876. *Proposed by Naoki Kimura, University of Washington.*

If a real valued function $f(x, y)$ of two real variables possesses all of its partial derivatives $\partial^{m+n}f(x, y)/\partial x^m \partial y^n$ at every point, is it necessarily continuous?

4877. *Proposed by O. P. Aggarwal and Irwin Guttman, University of Alberta*

Show that

$$\int_0^a e^{-t^{1/2}}dt = (\pi/2 - aI)^{1/2},$$

where

$$I = \int_{a^2}^{2a} \frac{e^{-\omega^{1/2}}d\omega}{\omega\sqrt{\omega - a^2}}.$$

4878. *Proposed by D. J. Newman, AVCO Research, Lawrence, Mass.*

A collection of closed bounded convex sets is given in the plane with the property that any three of them have a point in common. Prove they all have a point in common.

4879. *Proposed by L. J. Wallen, Massachusetts Institute of Technology*

Let f be a measurable real function defined for all real x, and let G be a continuous real function of two variables. Show that if

$$|f(x + y)| \leqq G(f(x), f(y)) \text{ for all } x \text{ and } y,$$

then f is bounded on bounded set.

4880. *Proposed by A. W. Goodman, University of Kentucky*

Show that for all α, β, $(0 < \alpha < \beta < \pi)$,

$$\int_0^\alpha \sqrt{\frac{\cos\theta - \cos\beta}{\cos\theta - \cos\alpha}}\,d\theta + \int_\beta^\pi \sqrt{\frac{\cos\beta - \cos\theta}{\cos\alpha - \cos\theta}}\,d\theta = \pi.$$

SOLUTIONS

Convergent Sequences

4828 [1959, 143]. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Do the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ converge, where

$$a_{n+1} = \int_0^1 \min(x, b_n, c_n)dx,$$

$$b_{n+1} = \int_0^1 \text{mid}\,(x, c_n, a_n)dx, \qquad c_{n+1} = \int_0^1 \max\,(x, a_n, b_n)dx,$$

and $\text{mid}(a, b, c) = b$ if $a \leqq b \leqq c$?

Solution by C. H. Cunkle, Cornell Aeronautical Laboratory, Buffalo, N. Y. The proposed sequences are convergent, with the limits 3/8, 1/2, 5/8, respectively.

Evidently $\min(x, b_n, c_n) \leqq x \leqq \max(x, a_n, b_n)$ so that, if $\text{mid}(x, a_n, c_n) = x$, we have

$$\min\,(x, b_n, c_n) \leqq \text{mid}\,(x, a_n, c_n) \leqq \max\,(x, a_n, b_n). \tag{1}$$

Now, if $\text{mid}(x, a_n, c_n) = a_n$, we have either $x \leqq a_n$ or $c_n \leqq a_n$, so that $a_n \leqq \max(x, a_n, b_n)$ implies (1) in this case also. A similar argument holds for $\text{mid}(x, a_n, c_n) = c_n$, so that (1) is true in all cases. By integration there results $a_{n+1} \leqq b_{n+1} \leqq c_{n+1}$, $n = 1, 2, \cdots$.

Now we have

$$a_{n+1} = \int_0^1 \min\,(x, b_n, c_n)dx \leqq \int_0^1 x dx = \tfrac{1}{2},$$

and similarly $c_{n+1} \geqq \frac{1}{2}$. Using this

$$b_{n+2} = \int_0^1 \text{mid}\,(x, a_{n+1}, c_{n+1})dx = \int_0^{1/2} \max\,(x, a_{n+1})dx + \int_{1/2}^1 \min\,(x, c_{n+1})dx$$

$$\leqq \int_0^{1/2} \frac{1}{2}\,dx + \int_{1/2}^1 x dx = 5/8.$$

Dually, $b_{n+2} \geqq 3/8$. Since $3/8 \leqq b_{n+2} \leqq c_{n+2}$, $a_{n+3} = \int_0^1 \min\,(x, b_{n+2}, c_{n+2})dx > 0$, and similarly $c_{n+3} < 1$.

It is now assumed that n is so large that $0 < a_n \leqq b_n \leqq c_n < 1$.

$$a_{n+1} = \int_0^{b_n} x dx + \int_{b_n}^1 b_n dx = \frac{2b_n - b_n}{2}$$

$$b_{n+1} = \int_0^{a_n} a_n dx + \int_{a_n}^{c_n} x dx + \int_{c_n}^1 c_n dx = \frac{a_n^2 - c_n^2 - 2c_n}{2},$$

$$c_{n+1} = \int_0^{b_n} b_n dx + \int_{b_n}^1 x dx = \frac{b_n^2 + 1}{2}.$$

Thus

$$b_{n+2} = \frac{1}{2}\left[\left(\frac{2b_n - b_n^2}{2}\right)^2 - \left(\frac{b_n + 1}{2}\right)^2 + 2\left(\frac{b_n + 1}{2}\right)\right]$$

$$= \frac{1}{2} + \frac{(2b_n - 1)(-2b_n^2 + 2b_n + 1)}{8} = b_n - \frac{(2b_n - 1)^3 + 5(2b_n - 1)}{16}.$$

Since $0 < -2b_n^2 + 2b_n + 1$ whenever $0 < b_n < 1$, either

$$\tfrac{1}{2} \leqq b_{n+2} \leqq b_n \quad \text{or} \quad \tfrac{1}{2} > b_{n+2} > b_n.$$

If follows that $\lim b_{2n} = \lim b_{2n+1} = \lim b_n = \frac{1}{2}$. Then

$$\lim a_n = \lim a_{n+1} = \lim \frac{2b_n - b_n^2}{2} = \frac{3}{8},$$

$$\lim c_n = \lim c_{n+1} = \lim \frac{b_n^2 + 1}{2} = \frac{5}{8}.$$

Also solved by J. W. Haake and by Y. Matsuoka.

Topological Transformations

4829 [1959, 147]. *Proposed by J. de Groot, University of Amsterdam, The Netherlands, and Purdue University*

Give a simple example of a continuum P such that, for every countable group G, there exists a group of topological transformations of P onto itself which is isomorphic to G.

Solution by J. V. Whittaker, University of British Columbia. For each integer $n > 0$, let $P_n = \{(r, \theta): 0 \leqq r \leqq 1, \theta = \pi/n\}$, where r, θ are polar coordinates in the plane. Also, let $P_0 = \{(r, \theta): 0 \leqq r \leqq 1, \theta = 0\}$, and $P = \bigcup_{n=0}^{\infty} P_n$. Evidently P is a one-dimensional continuum in the relative Euclidean topology. If G is a countable group, then G is isomorphic to a group of permutations of the positive integers, and we shall identify G with the latter group. For each $g \in G$, we define a corresponding mapping h of P as follows: $h(r, \pi/n) = (r, \pi/g(n))$ for $0 \leqq r \leqq 1$ and $n \geqq 1$, $h(r, 0) = (r, 0)$. Then h is a homeomorphism of P, for h and h^{-1} are clearly open at all points of P_n $(n \geqq 1)$; as neighborhoods of a point $(r_0, 0)$ $(0 < r_0 \leqq 1)$, we can choose the intersection of the set $\{(r, \theta): r_0 - \epsilon < r < r_0 + \epsilon\}$ with all but a finite number of the P_n $(n \geqq 1)$, and the images under h and h^{-1} of such sets have the same form. The correspondence $g \leftrightarrow h$ is evidently one-to-one. If $g_1 \leftrightarrow h_1$ and $g_2 \leftrightarrow h_2$, then $h_1 h_2(r, \pi/n) = h_1(r, \pi/g_2(n)) = (r, \pi/g_1 g_2(n))$ for all $n \geqq 1$, and $g_1 g_2 \leftrightarrow h_1 h_2$. Hence G acts effectively on P.

Also solved by Helen C. Arens, R. D. Gordon, and the proposer.

Angles in a Linear Normed Space

4831 [1959, 147]. *Proposed by J. L. Massera and J. J. Schäffer, Institute of Mathematics and Statistics, Montevideo, Uruguay*

J. A. Clarkson [*Uniformly convex spaces*, Trans. Amer. Math. Soc., vol. 40, 1936, pp. 396–414] introduced the following expression for the "angle" of two

nonvanishing elements of any linear normed space:

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Prove that if $x, y, x+y \neq 0$, then $\alpha[x+y, x] \leq \alpha[x, y]$.

Solution by Michael Golomb and R. M. Warten, Purdue University. To show

$$\left\| \frac{x}{\|x\|} - \frac{x+y}{\|x+y\|} \right\| \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|,$$

where $x, y, x+y \neq 0$; *put* $x/\|x\| = X$, $y/\|y\| = Y$,

$$\frac{x+y}{\|x+y\|} = Z, \qquad \frac{x+y}{\|x\|+\|y\|} = z, \qquad \frac{\|y\|}{\|x\|+\|y\|} = t.$$

Note $0<t<1$. Then $\|X\| = \|Y\| = \|Z\| = 1$, $\|z\| \leq 1$, $z = \|z\| Z$,

$$\|Z - z\| = 1 - \|z\| = \|Y\| - \|z\| \leq \|Y - z\|, \qquad (1-t)X + tY = z.$$

Hence $\|X-Z\| \leq \|X-z\| + \|Z-z\| \leq [t+(1-t)]\|X-Y\| = \|X-Y\|$, as required.

Also solved by R. D. Gordon, H. L. Loeb, the proposers, and one who signs himself "A. Polter Geist."

Diophantine Equations

4832 [1959, 147]. *Proposed by A. Oppenheim, University of Malaya, Singapore*

Find all integral solutions of the following Diophantine equations

(A) $$ax^2 + ay^2 + z^2 - 2axyz - 1 = 0 \qquad (a = 1, 2, \cdots),$$

(B) $$9x^2 + 25y^2 + 49z^2 - 210xyz - 1 = 0,$$

(C) $$9x^2 + 25y^2 + 4z^2 - 60xyz - 1 = 0.$$

The case $a=1$ of (A) is known.

Solution by D. C. B. Marsh, Colorado School of Mines. These equations are essentially the same in form as problem 4674 [1957, 121–122], and admit the same treatment.

1. Let $a = bc^2$ with b square-free. Let x, y, z be integers satisfying (A). Then we write

$$[z - b(cx)(cy)]^2 = [b(cx)^2 - 1][b(cy)^2 - 1]$$

which is equivalent to

$$b(cx)^2 - 1 = pq^2, \qquad b(cy)^2 - 1 = pr^2, \qquad z - b(cx)(cy) = pqr,$$

with p, q, r integers, p square-free and prime to a. It follows that integers $t = bcx$ and u exist such that

(1) $$t^2 - (bp)u^2 = b.$$

Conversely, let t_i, u_i and t_j, u_j be any pair of solutions of (1) having the t's multiples of bc, and set

$$x = t_i/bc, \qquad y = t_j/bc, \qquad z = t_it_j/b + pu_iu_j.$$

That these values satisfy (A) is directly verified. In this, p is not arbitrary but must be chosen so that (1) admits solutions. A necessary condition is that a be a quadratic residue of p; a sufficient condition is that $(p+1)/a$ be a square. The determination of all solutions of the "Pell equation" is a classical problem in the theory of quadratic forms.

2. Rewrite (B) as $[7z-(3x)(5y)]^2=[(3x)^2-1][(5y)^2-1]$ whence, with p again square-free and prime to 15,

$$(3x)^2 - pq^2 = 1, \qquad (5y)^2 - pr^2 = 1, \qquad 7z = 15xy + pqr.$$

$t^2-pu^2=1$ has solutions with $t\equiv 0$ both modulo 3 and modulo 5 if and only if the fundamental positive solution, t_1, is divisible by 15, in which cases all $t_{2j-1}\equiv 0 \pmod{15}$. For such, $x=t_{2i-1}/3$ and $y=t_{2j-1}/5$, $7z=t_{(2i-1)\pm(2j-1)}$. Thus all solutions of (B) are obtained from those p and t such that $t^2-pu^2=1$ has $t_1\equiv 0 \pmod{15}$ and some $t_{2j}\equiv 0 \pmod 7$. (There are solutions, *e.g.*, $x=10$, $y=6$, $z=257$.)

3. Proceeding with (C) as with (B), one needs $(2z)=t_{(2i-1)\pm(2j-1)}$, but no such t's are ever even, hence (C) has no solutions.

Also partially solved by C. C. Yalavigi.

Coefficients of a Polynomial

4833 [1959, 147]. *Proposed by F. H. Northover, Carleton University, Ottawa*

In the polynomial $A(x)=\sum_{i=0}^{n} a_ix^i$, $a_i\geqq 0$, $A(1)=1$, and the function

$$\frac{A'(1) - xA'(x)}{(1-x)\{1 - A(x)\}}$$

is expanded into $\sum_{i=0}^{\infty} b_ix^i$.* Prove $b_i\geqq A'(1)$ for all i.

Solution by Leonard Carlitz, Duke University. It is evidently necessary to assume $a_0<1$, for otherwise $A(x)=1$. Put

$$\frac{A'(1) - xA'(x)}{(1-x)(1-A(x))} - \frac{A'(1)}{1-x} = \frac{A'(1)A(x) - xA'(x)}{(1-x)(1-A(x))} = \sum_{r=0}^{\infty} c_rx^r;$$

it will suffice to show that $c_r\geqq 0$ for all r. We have

$$A'(1)A(x) - xA'(x) = \sum_{r=0}^{n} \{A'(1) - r\}a_rx^r.$$

* The upper limit is ∞, not n as originally printed.

If we put

$$\frac{A'(1)A(x) - xA'(x)}{1-x} = \sum_{r=0}^{n-1} d_r x^r,$$

then it follows that

$$\begin{aligned} d_k &= \sum_{r=0}^{k} \{A'(1) - r\} a_r \\ &= \sum_{r=0}^{k} a_r \sum_{s=0}^{n} s a_s - \sum_{r=0}^{k} r a_r \sum_{s=0}^{n} a_s \\ &= \sum_{r=0}^{k} \sum_{s=0}^{k} (s-r) a_r a_s + \sum_{r=0}^{k} \sum_{s=k+1}^{n} (s-r) a_r a_s \\ &= \sum_{r=0}^{k} \sum_{s=k+1}^{n} (s-r) a_r a_s, \end{aligned}$$

so that $d_k \geqq 0$ for $0 \leqq k \leqq n-1$. Since we have $1-A(x) = 1-a_0-A_1(x)$, where $A_1(x) = \sum_{r=1}^{n} a_r x^r$, then

$$\frac{1}{1-A(x)} = \frac{1}{1-a_0} + \frac{A_1(x)}{(1-a_0)^2} + \cdots.$$

It follows immediately that

$$c_r \geqq \frac{A'(1)}{1-a_0} \qquad (r = 0, 1, 2, \cdots),$$

which is slightly stronger than the stated result.

Also solved by R. D. Gordon, H. O. Pollak, C. C. Yalavigi, and the proposer.

Reduced Residue Systems

4834 [1959, 238]. *Proposed by Oystein Ore, Yale University*

It is well known that the number 30 is the largest integer such that the set of reduced residues (mod 30) includes no composite numbers. Determine all integers n such that the $\phi(n)$ reduced residues (mod n) are powers of primes.

Solution by N. J. Fine, The Institute for Advanced Study. Let the primes be $p_1 = 2, p_2, \cdots$ and let q_1, q_2 be the first two primes which do not divide n. A necessary and sufficient condition that n be of the required type is that $q_1 q_2 > n$. Each of the primes less than q_2 and different from q_1 divides n, and so does their product. Therefore the product of all primes less than q_2 does not exceed $q_1 n \leqq q_1^2 q_2$. If $q_2 = p_k$ then $p_1 p_2 \cdots p_{k-1} < p_{k-1}^2 p_k$. We shall show that this inequality is violated for all $k \geqq 6$. For if $p_1 p_2 \cdots p_{k-1} \geqq p_{k-1}^2 p_k$ for some $k \geqq 6$, then $p_1 \cdots p_k \geqq p_{k-1}^2 p_k^2$. Now it is known (Bertrand, Tchebysheff) that $p_j > \frac{1}{2} p_{j+1}$, so

$$p_1 \cdots p_k \geqq p_{k-1}^2 p_k^2 > p_k^2 p_{k+1}^2/16 > p_k^2 p_{k+1}$$

provided that $p_{k+1} > 16$, which is equivalent to $k \geqq 6$. But from $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 > 11^2 \cdot 13$ the inequality is violated for $k = 6$ and therefore for all $k \geqq 6$. Hence $p_k = q_2 \leqq 11$, $q_1 \leqq 7$, and $n < q_1 q_2 \leqq 77$. Examination of the numbers less than 77 quickly determines the following set:

(1), 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 18, 20, 24, 30, 42, 60.

Also solved by Jean M. Calloway, J. W. Ellis, Fritz Herzog, J. H. Hodges, D. C. B. Marsh, Georgia C. Smith, and Robert Spira.

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All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Les Limitations Internes des Formalismes. Par Jean Ladrière. (Collection de logique mathématique, Serie B, II) E. Nauwelaerts, Louvain, and Gauthier-Villars, Paris, 1957. xiii+715 pp. 650 frs. (Belgian), 7000 frs. (French). (About $14.00.)

This book is a rather complete compendium and exposition of modern work in the logical foundations of mathematics. Philosophical material is held to a very reasonable compass. By the "internal limitations" of the formalist method (axiomatic method which includes the necessary logic) the author has in mind results of the following kinds: (1) Gödel's incompleteness theorem to the effect than any "sufficiently rich" formal system must, if consistent, have undecidable sentences (sentences A such that neither A nor not-A are provable in the system), (2) Gödel's second theorem to the effect that the sentence in the system expressing the consistency of such a system is unprovable, (3) Church's theorem concerning the nonsolvability of the decision-problem for validity in the predicate calculus of the first-order, (4) results on models of formal systems ranging from that of Löwenheim-Skolem (any consistent first-order system has a denumerable model) to that of Rosser-Wang (a certain version of set-theory, if consistent, has no "standard" models). These results are but a few of the topics covered in this lengthy essay which, discounting notes, appendices, indices, tables, etc., runs to 400 pages of text. The reviewer believes that a newcomer to the field will find it a helpful guide.

THEODORE HAILPERIN
Lehigh University

Elementary Statistical Physics. By C. Kittel. Wiley, New York, 1958. ix+228 pp. $8.00. (College edition, $6.75.)

Although statistical mechanics is one of the most important and beautiful parts of theoretical physics, students of physics have been handicapped by the fact that most recent textbooks in the field have been written largely for the use of chemists.

The present book contains an excellent selection of topics of special importance to physicists. Other topics, such as phase equilibria, imperfect gases, and cooperative phenomena, that either are mainly of interest to chemists or can be treated only by elaborate special methods, are omitted.

Part I, comprising about half of the book, is devoted to statistical mechanics. The treatment is based on Gibbs' method of ensembles. Part II treats fluctuations, noise, and irreversible thermodynamics. Part III gives an introduction to kinetic theory and transport problems. The method of steepest descent and other matters largely of mathematical interest are outlined in four appendices. A set of problems is given at the end of most sections.

The treatment is fresh and lucid throughout and very concise. Its brevity, however, is effectively offset by the insertion of references after the headings of most sections. Many of these references are to very recent publications, and Appendix C, dealing with the solution of problems in molecular dynamics by electronic computers, is based on work not yet published.

J. Rud Nielsen
The University of Oklahoma

Contributions to the Theory of Nonlinear Oscillations. Vol. IV. S. Lefschetz, Ed. Annals of Mathematics Studies No. 41. Princeton University Press, Princeton, N. J., 1958. vii+211 pages. $3.75 (paperbound).

The ten papers by eleven authors which are collected in this volume show again, as the preceding three volumes did, the great width and interest of the work which is being done on nonlinear ordinary differential equations, as well as the great variety of methods which have been used successfully to attack the vast body of problems in this field.

The contributors to this volume are: S. Kakutani and L. Markus, S. Lefschetz, D. Bushaw, R. DeVogelaere, D. L. Slotnick, W. T. Kyner, G. Seifert, H. A. Antosiewicz, P. Mendelson, and R. W. Bass. The preface, by S. Lefschetz, contains excellent brief characterizations of the ten papers.

Anybody interested in the field of nonlinear differential equations will read this volume with great profit.

Oswald Wyler
University of New Mexico

Multivariate Correlational Analysis. By Philip H. DuBois. Harper, New York, 1958. xv+202 pp. $5.00.

Multivariate Correlational Analysis is a book without substance. It is largely dealing with descriptive statistics that are poorly explained and presented and computational methods that, while satisfactory for computing values of statistics, are not suitable for modern methods of analysis. The title of the book is misleading in that it does not cover the range of methods of multivariate correlational analysis, but only two topics, linear multiple regression with attendant simple, multiple, partial and part correlations and some concepts of factor analysis.

Chapters 1 to 10 and 13 and 14 deal with the correlations associated with multiple linear regression. The concepts of population and mathematical models are omitted. Notation departs from accepted statistical usage (*e.g.*, V for variance, $\sigma = \sqrt{V}$, C for covariance). All variances (total and residual) are obtained using division by sample size. The concept of degrees of freedom is ignored throughout the book. It is not recognized that curvilinear regression derives from the same formulas as multiple linear regression under usual models of statistics. Analyses are based on z-scores and, while this is sometimes useful, it is not helpful in regression analysis. Sampling distributions enter only in Chapter 14. Then the author makes statements regarding variances of predicted values that are incorrect. Only the large-sample test of a correlation is proposed and then only when the population correlation is zero. The exact distribution under that situation is not given nor is the z-transformation of Fisher discussed.

Chapters 11 and 12 purport to deal with factor analysis. Chapter 11 is an attempt to regard regression analysis as an introduction to factor analysis. By the author's own admission, the discussion of factor analysis is incomplete and is to be regarded as an attempt at a demonstration of relationships between factor analysis and partial correlation. The demonstration doesn't quite come off.

In summary, we cannot recommend this book for either a text or a reference book. We refer the reader interested in regression analysis to other books on the subject, for example, *Statistical Theory in Research* by R. L. Anderson and T. A. Bancroft or for more correlation analysis to *Statistical Inference* by H. M. Walker and Joseph Lev.

R. A. Bradley
Virginia Polytechnic Institute

Computability and Unsolvability. By Martin Davis. McGraw-Hill, New York, 1958. xxv+210 pp. $7.50.

The problems of finding decision methods in logic and the analogue in mathematics of finding algorithms were advanced significantly by Gödel's arithmetization procedures and Church's and Kleene's work in the theory of recursive functions. The goal of logical investigations relative to the decision problem is to find a mechanical method of deciding whether or not a proposition is true. In conse-

quence, if the word "computation" is used in a sufficiently wide sense, the decision problem as well as the algorithm problem can be classified as computational. The problem of computability, in this sense, then, is that of determining effective mechanical procedures for solving various types of problems. Moreover, if the procedures are to be purely mechanical, one ought to be able to construct a machine (computer) that could perform the computations. This suggests the connection of the problem of computability with that of the construction of computers, and further makes it possible to define "decidability," i.e. computability, in terms of the construction of such computers. Evidently, Gödel's incompleteness theorem could then imply that there were questions such that no machine could be so programmed that the questions could be decided.

The importance of the theory of recursive functions for the problem of the classification of computers is obvious. A machine can operate only in a fashion analogous to the process of building functions recursively. The output of the machine is the analogue of the decision.

Professor Davis has tried to give us "an introduction" to the theory of computability. The book first considers the general theory, discussing computable functions and recursive functions and relating these notions to the Turing machine. The author then considers some applications of the general theory to combinatorial problems, diophantine equations and mathematical logic. The final part of the book is devoted to some further aspects of the general theory, considering the "Kleene hierarchy," computable functionals and some results on unsolvable decision problems.

Although it is good to have this book, it is not precisely an "introduction" in the sense that students are led into the field "from scratch."

Louis O. Kattsoff
Boston College

Multivalent Functions. By W. K. Hayman. Cambridge Tract No. 48. Cambridge University Press, New York, 1958. viii+151 pp. $4.50.

This book presents an introduction to and a survey of certain aspects of the theory of multivalent functions mostly related to the recent developments of the author. He presupposes a knowledge of complex variables as given in Ahlfors, *Complex Analysis*, and of Lebesgue integration as given in Titchmarsh, *Theory of Functions*. The material is well organized and clearly written, containing complete proofs of all theorems but no exercises. The text makes frequent references to original sources and to related topics in the literature.

Various classes of multivalent functions are studied; for example, p-valent functions, really mean p-valent functions, and circumferentially mean p-valent functions. The book begins with the "classical" distortion theorems and coefficient inequalities for univalent functions, presented first in the usual way and then in a way which allows its extension to circumferentially mean p-valent functions. The bounds for $|f(z)|$ when f is areally p-valent are based primarily upon an area-length principle of the Ahlfors type. The distortion theorems and

coefficient inequalities for circumferentially mean p-valent functions depend upon the theory of symmetrization of Steiner, Polya, and Szego, a discussion of which is included in Chapter 4. The book closes with a presentation of the Löwner theory and its applications to univalent functions.

GEORGE SPRINGER
University of Kansas

Trigonometry. By Dorothy Rees and Paul K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1959. xi+318 pp. $3.96.

The authors state in their preface: "We try to maintain a proper balance between analytical and numerical work but lean somewhat toward the analytic side." The law of tangents, the half angle formulas, area of a triangle, and significant figures (but not slide rule) are discussed. Analytical and numerical chapters usually alternate; the exercise sets are a little shorter than usual (typically 24 problems, but see (1) below). There is almost no review of algebraic concepts; the definition of a periodic function is poorly stated and the definition of amplitude appears to have been inserted in a footnote as an afterthought, although the concept figures prominently in the exercises. The appearance of the pages is very good; the last two chapters give an introduction to complex numbers and spherical trigonometry; the book includes tables and answers to odd numbered exercises. Some unusual features of the book are: (1) Problem sets are broken up into groups of 4 very similar problems, so if there are 24 problems in a set, there are only 6 essentially different ones. (2) Each chapter has a summary, a review set of exercises, and a test-yourself set. (3) Thirteen photographs with headings show applications of trigonometry. These range from a collection of old musical instruments (sound waves show various sine curves) to an atomic electric plant (complex numbers are used to study alternating current).

B. H. ARNOLD
Oregon State College

Functions of Complex Variables. By Philip Franklin. Prentice Hall, Englewood Cliffs, N. J., 1958. ix+246 pp. $6.95.

This excellent work on the classical properties of the analytic functions of a complex variable was evidently designed to serve two purposes: (1) to provide mathematics majors at the advanced undergraduate level or the lower graduate levels an introduction to the specific subject treated and perhaps to analysis in general, and (2) to provide for physicists and engineers a basic text and a reference work on the fundamentals of the subject. In the reviewer's opinion it is admirably suited to both of these programs.

The exposition is rigorous but not tiresome, and suitably modern but definitely not pedantic. Apparently, this book was very carefully composed and its structure is such that the beginner can attain a solid knowledge of the subject by working through a series of stages (81 articles in all) of fairly uniform and moderate difficulty. There are a few subtleties that the novice may miss but if the

reader is ready for the subject and willing to study carefully he will be well rewarded for his effort. There are numerous misprints but they are obvious and easy to correct.

Each chapter begins with a brief sketch of the path to be followed and is broken up into short articles (usually one or two pages in length) interspersed with sets of problems—40 sets with a total of 645 problems. The problems are well graded and serve to emphasize the cardinal points and extend the theory. Among the subjects treated are: products, quotients, powers, roots, continuity, derivatives, Cauchy-Riemann equations, Laplace's equation, infinite series, power series, Riemann surfaces, conformal transformations, linear and bilinear transformations, Cauchy's integral theorem, the Cauchy-Goursat integral theorem, multiply connected regions, the Cauchy integral formula, Taylor's expansion, Morera's theorem, the maximum principle, Liouville's theorem, analytical continuation, Laurent's series, residues, and Schwarz-Christoffel transformations.

HOMER V. CRAIG
Boeing Airplane Company and
The University of Texas

BRIEF MENTION

Mathematics Dictionary. Edited by Glenn James and Robert C. James. 2nd ed. Van Nostrand, Princeton, N. J., 1959. 546 pp. $15.00.

The new revised edition of the James and James *Mathematics Dictionary* is certainly one of the most welcome publications in a long time, particularly since coverage of modern mathematical concepts has been greatly extended. Perhaps the most welcome addition to the book is the multilingual index which gives the English equivalents of mathematical words in Russian, German, French, and Spanish—a most welcome addition. This reviewer feels that even if you already possess the 1949 edition, the second edition represents sufficient expansion and extension to merit replacement of the earlier volume.

Siam Review. Edited by Robert J. Wisner. A Publication of the Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1959. 87 pp. $5.00 per year.

It is a pleasure to welcome this new publication to the growing list of mathematical journals. Volume 1, Number 1 begins with an excellent article, "Systems of Linear Relations," by R. A. Good, and continues to maintain interest throughout the volume. The editor states that although only two issues are planned for 1959, four issues will be forthcoming in 1960, and the price will then be $10.00 per year.

Experimental Music. By Lejaren A. Hiller, Jr. and Leonard M. Isaacson. McGraw-Hill, New York, 1959, vi+197 pp. $6.00.

If you are interested in the possibility of programming a modern computer to compose music, that is, to select ordered sets of notes in accordance with the rules of harmony and counterpoint, this is probably the book for which you are waiting.

Portfolio Selection. By Harry M. Markowitz. Wiley, New York, 1959. x+344 pp. $7.50.

This publication of the Cowles Foundation for Research in Economics is devoted

to the use of computing machines in the selection of an investment portfolio in which sufficient diversification is sought, taking into account factors such as: likely income, and appreciation, the uncertainty of income and appreciation, and the degree to which various securities tend to rise and fall together. A series of mathematical appendices bridge the gap with which the nonmathematical economist would be confronted.

Numerical Solution of Differential Equations. By William Edmund Milne. Wiley, New York, 1953. xi+275 pp. $7.25.

Computer laboratories will certainly welcome this second printing of Milne's excellent work.

Numerical Mathematical Analysis. By James B. Scarborough. 4th ed. Johns Hopkins, Baltimore, 1958. xxi+576 pp. $6.00.

The principal changes in the fourth edition of Professor Scarborough's work, originally published in 1930, appear in Graeffe's method and in the numerical solution of ordinary differential equations. A section on the smoothing of experimental data has also been added.

Plastic and String Models, Industrial Research Laboratories, Box 471, Hempstead, New York.

This is a collection of well-made, attractive, expensive ($11.75 to $18.75 each) models of plexiglass and elastic nylon string. They are of the same general nature as students have been making for years, showing the standard quadric and quartic surfaces, helicoids, cones, and intersecting cylinders; however, these are particularly well-made and may be of interest to mathematicians who are asked to recommend a source for mathematical models.

Scientific American Books. Simon and Schuster, New York, 1955–57. $1.45 each.

The Universe. xvi+142.
Plant Life. xiii+237.
The Physics and Chemistry of Life. xi+272.
The Planet Earth. viii+168.
Twentieth-Century Bestiary. xi+243.

Although not mathematical, this little series provides excellent source books for current scientific articles, presented in concise carefully chosen phrases.

Symbolic Logic and the Game of Logic. By Lewis Carroll. Dover, New York, 1958. xxxi+96 pp. $1.50.

Lewis Carroll's writings on mathematical recreation are never old.

The Dynamics of Particles. By Arthur Gordon Webster. 2nd ed. Dover, New York, 1959. xi+588 pp. $2.35.

This is a reprint of the second (1912) edition of Webster's 1904 book.

Computational Methods of Linear Algebra. By V. N. Faddeeva, Translated by Curtis D. Benster. Dover, New York, 1959. x+252 pp. $1.95.

Congratulations to Dover on the timely appearance of this volume devoted to matrix computation. It is a methods book, not a theory book, and a most welcome one.

A Course of Pure Mathematics. By G. H. Hardy. 10th ed. Cambridge University Press, New York, 1959. xii+509 pp. $3.75.

A paperbacked edition of Hardy's well-known book.

Meson Physics. By Robert E. Marshak. Dover, New York, 1958. viii+378 pp. $1.95.

A reprint of Marshak's 1952 book based on his lectures at the University of Rochester and Columbia University in 1950.

The Measurement of Power Spectra from the Point of View of Communications Engineering. By R. B. Blackman and J. W. Tukey. Dover, New York, 1959. x+190 pp. $1.85.

A well-written modern book on applied Fourier analysis reprinted from the 1958 *Bell System Technical Journal.*

Theory of Functionals and of Integral and Integro-Differential Equations. By Vito Volterra. Dover, New York, 1959. 226 pp. $1.75.

A reprint of Volterra's 1930 book describing his 1925 lectures at The University of Madrid, with a list of publications and a 28-page biography appended.

Pillow Problems and *A Tangled Tale.* By Lewis Carroll. Dover, New York, 1958. xx+152 pp. $1.50.

Two more welcome reprints.

Trigonometric Series, Vol. 1 and 2. By A. Zygmund. 2nd ed. Cambridge University Press, New York, 1959. xii+383 pp., vii+354 pp. $15.00 each, $27.50 set.

This is a complete revision and up-to-date extension and modernization of Professor Zygmund's classic. It has more than doubled in size. Volume 1 contains the completely rewritten material from the original work on trigonometric and Fourier series. Volume 2 provides much material previously unpublished in book form, and covers trigonometric interpolation, differentiation of series, generalized derivatives, interpolation of linear operations, and additional information about Fourier coefficients, including the Littlewood-Paley function. Each volume concludes with chapter notes and Volume 2 also contains excellent bibliography.

Group Theory. By Eugene P. Wigner. Translated from the German by J. J. Griffin. Academic Press, New York, 1959. xi+372 pp. $8.80.

Both physicists and mathematicians will certainly welcome this careful translation of Wigner's original classic on the application of group theory to quantum mechanics and electron spin theory. Three new chapters have been added to the translation on Rachah coefficients, time inversion, and on the limits of representation coefficients.

Plane Trigonometry. By Raymond W. Brink. 3rd ed. Appleton-Century-Crofts, New York, 1959. xii+228+110 pp. $4.00.

This well-known book needs no review other than notice that a new edition has appeared. This edition retains the flavor of its predecessors.

Mathematical Puzzles and Pastimes. By Aaron Bakst. Van Nostrand, Princeton, N. J., 1954, vi+206 pp. $4.00.

A new printing of an old favorite which belongs on every shelf of recreational mathematics.

Reflections of a Mathematician. By L. J. Mordell. Canadian Mathematical Congress, Montreal, Canada, 1959. vii+50 pp. $1.00.

This heart-warming book is the outgrowth of an after-dinner speech given to the Royal Society of Canada in 1955. It should, in my opinion, be read by every budding mathematician as well as savored by those already past the budding stage. Taste, insight, and care have gone into its preparation.

Foundations of Combinatorial Topology. By L. S. Pontryagin. Graylock, Rochester, N. Y., 1952. xii+99 pp. $4.00.

This translation from the Russian by F. Bagemihl, H. Komm, and W. Seidel deserves considerably more recognition that it has been receiving. If you are seeking information at an elementary level on Betti groups, continuous mappings and fixed points, this may well be the volume you desire.

Foundations of the Nonlinear Theory of Elasticity. By V. V. Novozhilov. Graylock, Rochester, N. Y., 1953. vi+233 pp. $4.00.

Another translation by Bagemihl, Komm, and Seidel which merits consideration.

Advanced Calculus. By Edwin Bidwell Wilson. Dover, N. Y., 1958. xi+566 pp. $2.45.

A paperback edition of the classical *Advanced Calculus* by Wilson which will be welcomed by many readers.

Mathematics for the Academically Talented Student in the Secondary School. Edited by Julius H. Hlavaty. National Education Association of the United States, Washington, D. C., 1959. 48 pp. Single copy $.60. Quantity discounts available.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the meeting in Chicago, Illinois, January 27–29, 1960 from 9 a.m. to 5 p.m. The Placement Service Desk will be on the third floor of the Conrad Hilton Hotel. There will be no charge for registering either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants on request.

Job applicants and employers wishing to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for forms which must be completed and returned to Providence not later than January 8, 1960, in order to be included in the listings of the Chicago meeting free of charge. Forms which arrive after the closing date, but before January 26, will be listed for a late registration fee of $3.00, and will also be included in the sold listings, but not until ten days after the meeting. Printed listings will be available for sale during and after the meeting.

It is essential that applicants and employers register at the Placement Service Desk promptly on arrival to receive the call number which has been assigned to them.

SUMMER EMPLOYMENT IN MATHEMATICS

It is planned to compile a listing of Summer Employment for Mathematicians and College Mathematics Students. Institutions and industrial firms who would welcome inquiries for summer employment for 1960 should write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, indicating the number of openings, description of positions, and officer to whom applications should be directed.

SIAM VISITING LECTURESHIPS

The Society for Industrial and Applied Mathematics announces the establishment of a Visiting Scientist Lectureship Program which will serve the dual purpose of familiarizing college and university groups with contemporary mathematical activity in applied and industrial settings, and of making available representative research mathematicians, active in areas of current interest, to industrial groups for visiting lectures and discussions.

A grant of the National Science Foundation supports the part of the program which is aimed at colleges and universities, making it possible to provide the broadest geographical coverage. The visiting scientists will be prepared to give general lectures, to discuss more specific topics with individual classes and seminars, to advise in individual or group conferences on questions of curriculum, research in progress, or opportunities and requirements in the field of applied mathematics, and generally to cooperate in every way with the colleges in furthering the aims of the program. A normal visit will last for two or three days. Contributions to the program will be sought from the institutions visited in the form of a portion of the traveling expenses and subsistence for the visitor while on campus, but no institution will be denied visits solely because it cannot afford to make such a contribution.

The complementary part of the program will provide opportunities for professional mathematicians who have gone into industry to maintain their ties with current phases of research mathematics. Formal lectures of a survey nature, seminar talks, and nonconsultative discussions of personal research interests will constitute the visitor's activity during his normally one or two day visit. It is expected that the inviting organization will in each instance bear the lecturer's costs in full.

The program is under the direction of Dr. F. J. Weyl of the Office of Naval Research, Washington 25, D. C., who is assisted by a committee of the Society appointed for this purpose. Information on the panels of lecturers and on application procedures will be available in the near future.

TEMPORARY MEMBERSHIPS FOR THE ACADEMIC YEAR 1960–61 INSTITUTE OF MATHEMATICAL SCIENCES, NYU

The Institute of Mathematical Sciences at New York University offers temporary memberships to mathematicians and other scientists holding the Ph.D. degree who intend to study and do research in the fields in which the Institute is particularly active. These fields include FUNCTIONAL ANALYSIS, FUNCTION THEORY, DIFFERENTIAL EQUATIONS, MATHEMATICAL PHYSICS, FLUID DYNAMICS AND MAGNETOHYDRODYNAMICS, ELECTROMAGNETIC THEORY and NUMERICAL ANALYSIS AND DIGITAL COMPUTING.

The temporary membership program is designed primarily to alleviate the present critical shortage of scientists trained in mathematical physics, applied mathematics, and mathematical analysis. The program is being supported by the National Science Foundation and also by funds contributed by industrial firms to New York University.

Temporary members may participate freely in the research projects, the advanced

graduate courses, and the research seminars of the Institute, and they will have the opportunity of using the computational facilities. The temporary members will receive a grant commensurate with their status. Temporary memberships are awarded for one year, but may be renewed in special cases. Appropriate arrangements can be made for applicants who expect to be on leave of absence from their institutions.

Requests for information and for application blanks should be addressed to the Membership Committee, Institute of Mathematical Sciences, 25 Waverly Place, New York 3, N. Y.

PERSONAL ITEMS

University of Alberta: Assistant Professors E. L. Whitney, G. C. Cree, and W. J. Bruce have been promoted to Associate Professors; Associate Professor Lee Lorch, Wesleyan University, has been appointed Associate Professor; Dr. J. R. McGregor, University of Birmingham, England, and Mr. R. W. Longley, Canadian Meteorological Service, have been appointed Assistant Professors.

Allegheny College: Dr. A. L. Rabenstein, U. S. Army Mathematics Research Center, University of Wisconsin, has been appointed Assistant Professor; Mr. G. S. Tsiang, Southern Illinois University, has been appointed Instructor.

American University: Mr. H. H. Chu, Oklahoma State University, and Mrs. Luceil K. Arnett have been appointed Instructors.

University of California, Los Angeles: Assistant Professor Barrett O'Neil has been promoted to Associate Professor; Associate Professor E. A. Coddington has been promoted to Professor; Drs. R. J. Blattner and H. J. Weinitschke have been promoted to Assistant Professors; Professor I. S. Sokolnikoff has been awarded a Guggenheim Fellowship for 1959–60 and will spend most of the year in Zurich; Professor P. B. Johnson, Occidental College, has been appointed Associate Professor; Dr. Leo Breiman has been appointed Assistant Professor; Dr. Tilla Klotz, New York University, has been appointed Instructor; Dr. Gunter Lumer, University of Chicago, has been appointed Acting Assistant Professor.

Case Institute of Technology: Mr. C. G. Cullen, Worcester Polytechnic Institute, has been appointed Instructor; Dr. W. J. Kammerer, University of Wisconsin, has been appointed Assistant Professor in the Operations Research Group; Professor R. C. Bose, on leave from the University of North Carolina, has been appointed Visiting Professor in Statistics; Dr. D. K. Ray-Chaudhuri has been appointed Research Associate in the Statistical Laboratory.

Colorado State University: Assistant Professor E. E. Remmenga has been promoted to Associate Professor; Assistant Professor R. H. Niemann, Worcester Polytechnic Institute, has been appointed Assistant Professor.

Dartmouth College: Professor Grace E. Bates, Mount Holyoke College, has been appointed Visiting Professor during 1959–60; Dr. V. E. Benes, Bell Telephone Laboratories, Murray Hill, New Jersey, has been appointed Visiting Lecturer during 1959–60; Dr. A. J. Fabens, Stanford University, has been appointed Research Instructor during 1959–60.

Georgetown University: Assistant Professor M. W. Oliphant has been promoted to Associate Professor; Dr. A. K. Aziz has been promoted to Assistant Professor; Mr. C. L. Strain, Purdue University, has been appointed Instructor.

Harvard University: Associate Professor J. T. Tate has been promoted to Professor; Professor Raoul Bott, University of Michigan, has been appointed Professor.

Hobart and William Smith Colleges: Professor W. H. Durfee was granted the honorary degree of Doctor of Science at the June 1959 Commencement and has retired as Professor Emeritus; Professor R. L. Beinert has been appointed Head of the Mathematics Department.

University of Idaho: Assistant Professor A. E. Labarre, Jr. has been promoted to Associate Professor; Mrs. Elna H. Grahn has been promoted to Assistant Professor; Visiting Associate Professor S. S. Shrikhande, University of North Carolina, has been appointed Associate Professor; Mr. Newman Fisher, San Francisco State College, has been appointed Instructor.

University of Illinois: Associate Professors C. R. Blyth, H. E. Vaughn, and Joseph Landin have been promoted to Professors; Assistant Professor Beulah Armstrong has been promoted to Associate Professor; Mr. C. A. Phillips has been promoted to Assistant Professor; Assistant Professor Pierre Samuel, University of Clermont-Ferrand, France, has been appointed Visiting Professor; Professor Paulo Ribenboim, Inst. de Math. Pura e Aplicada, Rio de Janeiro, Brazil, has been appointed Visiting Associate Professor; Professor S. S. Cairns is on leave of absence at the Institute for Advanced Study; Assistant Professor N. T. Hamilton is on leave of absence at Johns Hopkins University; Drs. Hiram Paley, University of Wisconsin, Herbert Wilf, Columbia University, R. L. Bishop, Roberto Diaz, and D. M. Roberts have been appointed Assistant Professors; Mr. J. J. Rotman, University of Chicago, has been appointed Research Associate.

Indiana University: Associate Professor George Whaples has been promoted to Professor; Assistant Professor A. H. Wallace, University of Toronto, has been appointed Associate Professor; Visiting Associate Professor Gopinath Kallianpur, Michigan State University, has been appointed Associate Professor.

State University of Iowa: Associate Professor H. A. Dye, University of Southern California, has been appointed Associate Professor; Dr. J. F. Jakobsen, University of Missouri, has been appointed Instructor.

Kansas State University: Associate Professor L. E. Fuller has been promoted to Professor; Miss Rochelle Abend, University of Illinois, has been appointed Instructor.

Louisiana Polytechnic Institute; Professor W. B. Temple has been appointed Head of the Department of Mathematics; Assistant Professor D. E. Johnson has been promoted to Associate Professor.

McMaster University: Assistant Professor J. H. H. Chalk has been promoted to Associate Professor; Dr. W. B. Pennington, University of London, England, has been appointed Visiting Professor for the 1959–60 academic year.

Miami University: H. S. Pollard is retiring as Chairman, but will continue to serve as Professor; Dr. R. G. Selfridge, Naval Ordnance Test Station, China Lake, California, has been appointed Associate Professor and Director of the Computing Center; Assistant Professor Kermit Hutcheson, Southern Technical Institute, has been appointed Instructor.

Michigan State University: Assistant Professors M. Isobel Blyth, H. E. Campbell, W. E. Deskins, and G. P. Weeg have been promoted to Associate Professors; Dr. O. E. Taulbee, Lockheed Aircraft Corporation, Marietta, Georgia, has been appointed Associate Professor.

University of Michigan: Associate Professors D. A. Darling, M. O. Reade, R. C. Lyndon, and C. L. Dolph have been promoted to Professors; Assistant Professors F. W. Gehring, Frank Harary, and A. L. Mayerson have been promoted to Associate Professors; Dr. Edward Halpern has been promoted to Assistant Professor; Dr. A. E. Heins, John Simon Guggenheim Memorial Foundation, Copenhagen, has been appointed Professor; Dr. Morton Brown and Dr. D. A. Jones, State University of Iowa, have been appointed Assistant Professors; Drs. M. S. Ramanujan, Muslim University, Hsin Chu, Taiwan Normal University, Taipai, Formosa, Gerald Hedstrom and R. H. Rosen, University of Wisconsin, and J. M. Kister, Midwestern University, have been appointed Instructors; Drs. Stanislaw Mrowka, Polish Academy of Sciences, Moseh Shimrat, Hebrew University, Jerusalem, Israel, and R. L. Schaetz, Institute of Technology, Munich, Germany, have been appointed Visiting Lecturers; Drs. Frederick Sleator, University of Michigan Research Institute, G. P. Patil, and Messrs. B. W. Arden,

Michigan Tabulating Service, and B. Stubblefield, have been appointed Lecturers.

University of Mississippi: Colonel H. L. Quarles has returned to the Department after his retirement from the Army and has been appointed Assistant Professor; Dr. C. W. Barnes, University of Massachusetts, has been appointed Associate Professor; Mrs. Corrie D. Quarles, a former department member, has been appointed Acting Assistant Professor.

University of Montreal: Dr. Roland Guy, Post-doctoral Fellow of the National Research Council at the University of Montreal, and Mr. Pierre Robert, Sun Life Assurance Company of Canada, have been appointed Assistant Professors.

North Carolina State College: Assistant Professors G. C. Caldwell, D. M. Peterson, and H. A. Petrea have been promoted to Associate Professors.

Ohio Wesleyan University: Dr. R. A. Roberts, Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania, has been appointed Associate Professor; Dr. A. A. Johnson, Case Institute of Technology, and Mrs. Emma D. Johnson, University of Akron, have been appointed Assistant Professors; Mr. G. M. Nielsen, University of Wisconsin, has been appointed Instructor.

University of Oklahoma: Associate Professor B. S. Whitney has been promoted to Professor; Assistant Professor C. A. Nichol, Illinois Institute of Technology, has been appointed Assistant Professor.

Rensselaer Polytechnic Institute: Assistant Professors W. E. Boyce and R. C. DiPrima have been promoted to Associate Professors.

University of South Carolina: Professor P. K. Smith, Louisiana Polytechnic Institute, has been appointed Professor; Assistant Professor Karl Matthies, University of Cincinnati, has been appointed Associate Professor.

University of South Dakota: Mr. W. E. de Malignon, University of Wisconsin, has been appointed Assistant Professor; Mr. T. A. Jenkins, State University of Iowa, has been appointed Instructor.

Stanford University: Assistant Professor John McGregor has been promoted to Associate Professor; Dr. J. W. Lamperti has been promoted to Assistant Professor; Associate Professor Robert Finn, California Institute of Technology, has been appointed Professor; Drs. Paul Fife, New York University, and Patrick Barry, Imperial College, London, England, have been appointed Instructors.

Texas Southern University: Mrs. Velma M. Williams, Wiley College, and Mr. Beverly Smith have been appointed Instructors.

Tufts University: Mr. Paul Aizley, University of Arizona, and Mr. Willard Draisin, Connecticut College, have been appointed Instructors.

University of Tulsa: Associate Professor W. A. Rutledge, University of South Carolina, has been appointed Professor; Mr. T. W. Cairns, Oklahoma State University, has been appointed Assistant Professor; Mr. J. T. Day, Chance Vought Aircraft, Dallas, Texas, has been appointed Instructor; Dr. R. A. Greenkorn has been appointed Lecturer.

U. S. Naval Academy: Assistant Professors E. G. Swafford, C. E. Thompson, and J. H. White have been promoted to Associate Professors; Messrs. C. L. Beall, Johns Hopkins University Applied Physics Laboratory, and W. K. Stahlman, Geneva College, have been appointed Assistant Professors.

Utah State University: Professor N. C. Hunsaker is on sabbatical leave during 1959–60 and will be at the University of Illinois on an NSF scholarship; Mr. W. L. Pope, Lockheed Aircraft Company, has been appointed Assistant Professor.

University of Utah: Associate Professor J. H. Barrett will be on leave during 1959–60, and will be Associate Professor at the Mathematics Research Center, U. S. Army, University of Wisconsin; Associate Professor B. W. Helton, Southwest Texas State College, has been appointed Assistant Professor.

University of Washington: Assistant Professor D. B. Dekker has been promoted to Associate Professor; Assistant Professor John Isbell has been promoted to Associate Professor and is on leave at Purdue University; Assistant Professor J. M. Kingston has been promoted to Associate Professor and is on leave at the University of California, Berkeley, on an NSF Fellowship; Assistant Professor H. S. Bear is on leave at Princeton University; Assistant Professor R. K. Getoor is on leave at Massachusetts Institute of Technology on an NSF Felloswhip; Professor A. G. Walker, University of Liverpool, England, has been appointed Visiting Professor; Professor Jun-iti Nagata, Osaka City University, Japan, has been appointed Visiting Associate Professor; Dr. Elmar Thomas has been appointed Visiting Professor on the Fulbright Program; Dr. Jun-ichi Hano, University of Chicago, and Assistant Professor Harry Corson, Tulane University, have been appointed Assistant Professors; Drs. G. J. Minty, Duke University, W. B. Woolf, University of Michigan, E. T. Kobayashi, and Mr. J. D. Reid, have been appointed Instructors.

Wesleyan University: Drs. J. J. McKibben and R. G. Long have been promoted to Assistant Professors; Professor Emeritus F. L. Griffin, Reed College, has been appointed Visiting Professor.

Western Illinois University: Dr. H. G. Ayre has been promoted to Dean of the School of Arts and Sciences and Professor of Mathematics; Associate Professor J. J. Stipanowich has been promoted to Professor and Head of the Mathematics Department; Mr. Jerry Shryock has been appointed Instructor.

University of Wichita: Mrs. Mary Staadt and Mr. E. L. Dubowsky have been promoted to Assistant Professors; Miss Jane Secrest, University of Kansas, has been appointed Instructor.

University of Wisconsin: Assistant Professor Edward Fadell has been promoted to Associate Professor; Mr. O. J. Marshall has been promoted to Assistant Professor; Visiting Professor Walter Rudin, Yale University, and Associate Professor K. T. Smith, University of Kansas, have been appointed Professors; Drs. Walter Littman, University of California, Berkeley, Hans Schneider, Queen's University, Northern Ireland, and Assistant Professor J. J. Andrews, University of Georgia, have been appointed Assistant Professors; Dr. Mary E. Rudin, Yale University, has been appointed Visiting Assistant Professor; Dr. Lida K. Barrett, University of Utah, has been appointed Visiting Lecturer; Dr. A. N. Feldzamen, Yale University, has been appointed Research Instructor; Professor R. C. Buck is on leave this year with Focus in Princeton.

Worcester Polytechnic Institute: Professor Harris Rice retired on June 30, 1959 with the title Professor Emeritus; Messrs. G. C. Branche, University of Rochester, Bernard Howard, Brown University, and Dr. Nosup Kwak, Duke University, have been appointed Instructors.

Dr. L. W. Akers has been appointed Assistant Professor of Hygiene and Physician to Student Health Service at the University of Illinois.

Assistant Professor A. G. Azpeitia, University of Massachusetts, has been promoted to Associate Professor.

Mr. J. E. Beam, Harvard University, has accepted a position as Staff Member, Physical Sciences, with the Sandia Corporation, Albuquerque, New Mexico.

Professor Lulu Bechtolsheim, University of Redlands, will be on sabbatical leave during 1959–60.

Mr. J. G. Beery, Sandia Corporation, has accepted a position as Research Assistant with the Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

Associate Professor J. H. Blau, Antioch College, is on sabbatical leave during 1959–60 and will be at Stanford University.

Associate Professor J. R. Blum, Indiana University, has accepted a position as Technical Staff Member with the Sandia Corporation, Albuquerque, New Mexico.

Dr. Louis Brickman, University of Pennsylvania, has been appointed Instructor at Yale University.

Assistant Professor W. E. Briggs, University of Colorado, has been promoted to Associate Professor and will be Acting Chairman of the Department during 1959–60.

Mrs. Helen L. Brooks, University of Toledo, has been promoted to Assistant Professor.

Assistant Professor F. E. Browder, Yale University, has been promoted to Associate Professor.

Mr. R. E. Cady, West Virginia University, has accepted a position as a Management Trainee with the American Steel and Wire Company, Cleveland, Ohio.

Dr. Daniel Cohen, University of Wisconsin, has been appointed Instructor at the University of Manchester, England.

Assistant Professor C. S. Coleman, Wesleyan University, has been appointed Assistant Professor at Harvey Mudd College.

Dr. R. C. Courter, University of Wisconsin, has been appointed Instructor at Rutgers University.

Associate Professor Geoffrey Crofts, Occidental College, has accepted a position as Actuarial Training Director of the Occidental Life Insurance Company, Los Angeles, California.

Dr. Yvonne H. Cuttle, University of Oregon, has been appointed Instructor at the University of Saskatchewan, Canada.

Assistant Professor D. F. Dawson, University of Missouri, has been appointed Assistant Professor at North Texas State College.

Dr. R. E. Dowds, Purdue University, has been appointed Associate Professor at Butler University.

Professor Martha E. Edwards, Campbell College, has been appointed Professor at Shorter College.

Mr. L. R. Espeland, University of Colorado, has been appointed Instructor in Iowa Falls Community Schools, Iowa.

Mr. H. L. Farris, Society of Exploration Geophysicists, Tulsa, Oklahoma, has been appointed Instructor at the Episcopal High School, Alexandria, Virginia.

Mr. D. O. Faus, Oregon State College, has been appointed a Mathematics Teacher at Pacific Grove High School, Pacific Grove, California.

Dr. C. F. Federspiel has been appointed Assistant Professor in the Department of Preventive Medicine, Vanderbilt University.

Dr. William Forman, Brooklyn College, has been promoted to Assistant Professor.

Professor Tomlinson Fort, University of South Carolina, will be Visiting Professor at Emory University during the academic year 1959–60.

Mr. Allen Fox, International Business Machines Corporation, Poughkeepsie, New York, has accepted a position as Electronic Systems Engineer with the System Development Corporation, Lodi, New Jersey.

Mr. J. B. Fraleigh, Dartmouth College, has been appointed Instructor at Mount Holyoke College during 1959–60.

Mr. W. G. Franzen, Illinois Institute of Technology, has been appointed Instructor at Louisiana State University.

Mr. H. P. Friedman, Bulova Research and Development Laboratories, Woodside, New York, has accepted a position as Mathematician with the System Development Corporation, Lodi, New Jersey.

Associate Professor K. D. Fryer, Royal Military College of Canada, has been appointed Professor at the University of Waterloo, Waterloo, Ontario.

Professor P. R. Garabedian, Stanford University, has been appointed Professor at New York University.

Dr. R. E. Gomory, Princeton University, has accepted a position as Manager of Business Systems Research with International Business Machines Corporation, Yorktown Heights, New York.

Professor M. O. Gonzalez, University of Alabama, has been appointed Professor at the University of Havana.

Professor Christopher Gregory, University of Hawaii, has been appointed Chairman of the Mathematics Department.

Dr. E. H. Hanson, Chance Vought Aircraft Inc., Dallas, Texas, has accepted a position as Chief of the Technical Staff, Land-Air, Inc., Point Mugu, California.

Mr. Paul Harms, Iowa State College, has been appointed Instructor at Simpson College.

Associate Professor C. A. Hayes, Jr., University of California, Davis, has been promoted to Professor.

Lt. S. P. Holzapfel, Nuclear Weather Training Center, San Diego, California, has accepted a position as Mathematician with Litton Industries, Inc., Beverly Hills, California.

Assistant Professor J. T. Humphrey, Grambling College, has been promoted to Associate Professor.

Mr. J. L. Hursch, Jr., University of California, Berkeley, has been appointed Assistant Professor at San Diego State College.

Dr. D. G. Johnson, Purdue University, has been appointed Assistant Professor at Pennsylvania State University.

Captain F. D. Johnson has been appointed Professor at Wentworth Military Academy and Junior College.

Mr. W. J. L. Kane, University of Pennsylvania, has been appointed a Teacher at West Hempstead Junior-Senior High School.

Mr. E. H. Kanning, III, Lockheed Aircraft Corporation, Sunnyvale, California, has accepted a position as Mathematician with Remington Rand Univac, St. Paul, Minnesota.

Professor Leo Katz, Michigan State University, has accepted the position of Scientific Liaison Officer with the Office of Naval Research, London, England.

Dr. Maurice Kennedy, Stanford University, has been appointed College Lecturer at University College, Dublin, Ireland.

Dr. H. S. Kieval, Polytechnic Institute of Brooklyn, has been appointed Associate Professor with the State University College of Education, New Paltz, New York.

Mr. D. B. Kirk, Curtiss-Wright, Quehanna, Pennsylvania, has accepted the position of Associate Research Mathematician with the Willow Run Laboratories, University of Michigan.

Assistant Professor J. E. Kist, Wayne State University, has been appointed Assistant Professor at Pennsylvania State University.

Dr. S. G. Kneale, General Electric Company, has accepted a position as Senior Staff Engineer with Avco-Crosley Division, Cincinnati, Ohio.

Mr. W. E. Kopka, International Business Machines Corporation, Poughkeepsie, New York, has accepted a position as Consultant with the Service Bureau Corporation, New York.

Associate Professor H. W. Kuhn, Bryn Mawr College, has been appointed Associate Professor at Princeton University and has resumed his part-time position as Executive Secretary, Division of Mathematics, National Academy of Sciences—National Research Council.

Mr. C. P. Lecht, Lincoln Laboratory, Massachusetts Institute of Technology, has accepted a position as Computer Systems Designer for the Mitre Corporation, Lexington, Massachusetts.

Mr. Harold Ledford, University of Kentucky, has accepted a position as Mathematician for the U. S. Government, Redstone Arsenal, Alabama.

Professor Walter Leighton, Carnegie Institute of Technology, has been appointed Elias Loomis Professor of Mathematics and Chairman of the Department at Western Reserve University.

Mr. L. A. Liddiard, University of Minnesota, has accepted a position as Associate Engineer with the Boeing Airplane Company, Pilotless Missile Division, Seattle, Washington.

Mr. M. F. Lipp, Stanford University, has accepted a position as Analyst with the Radio Corporation of America, Moorestown, New Jersey.

Associate Professor D. B. Lloyd, District of Columbia Teachers College, has been appointed Chairman of the Division of Mathematics.

Associate Professor A. J. Lohwater, University of Michigan, has been appointed Associate Professor at the Rice Institute.

Mr. Peter Longley, University of Utah, has been appointed Instructor at the University of Alaska.

Dr. R. D. Luce, Harvard University, has been appointed Professor of Psychology at the University of Pennsylvania.

Dr. W. S. Mahavier, Illinois Institute of Technology, has been appointed Assistant Professor at the University of Tennessee.

Mr. R. D. Mason, Jr., Burroughs Corporation, Paoli, Pennsylvania, has been appointed Research Assistant in the Operations Research Department of Johns Hopkins University, Bethesda, Maryland.

Professor Emeritus R. B. McClenon, on leave from Grinnell College, will be Visiting Professor for one year at the University of Corpus Christi.

Mr. J. P. McGay, University of Tulsa, has been appointed Vice-President of the Raydean Company, Inc., Tulsa, Oklahoma.

Lt. W. M. McKeeman, United States Navy, has been appointed Instructor at the United States Naval Academy.

Mr. F. L. McMains, Jr., University of Arizona, has been appointed Instructor at the University of Oregon.

Mr. R. E. Metcalf, Jr., Portland State College, has accepted a position as Physicist with the Naval Ordnance Test Station, China Lake, California.

Mr. H. C. Miller, Jr., University of Alabama, has been promoted to Assistant Professor.

Mr. J. A. Minahan has accepted a position as Physicist with the Sprague Electric Company, North Adams, Massachusetts.

Dr. H. J. Miser, United States Air Force, has been appointed Head of the Operational Sciences Laboratory of the Research Triangle Institute, Durham, North Carolina.

Mr. J. E. Mueller, University of Illinois, has been appointed Chaiman of the Department of Mathematics at Greenwich High School, Greenwich, Connecticut.

Mr. W. T. Neis, Bureau of the Census, Suitland, Maryland, has accepted a position as Mathematical Statistician with the Naval Ammunition Depot, Quality Evaluation Laboratory, Lualualei, Oahu, Hawaii.

Mr. M. K. Nestell, University of Wisconsin, has been appointed Instructor at Southern Missionary College.

Assistant Professor Andrewa R. Noble, Chico State College, has been appointed Associate Professor and Chairman of the Mathematics Department at Pacific University.

Mr. J. T. Parent, University of Maine, has been appointed Instructor at Dutchess Community College.

Assistant Professor W. M. Perel, Texas Technological College, has been appointed Associate Professor at Louisiana State University, New Orleans.

Dr. R. S. Pinkham, Princeton University, has been appointed Associate Professor at Rutgers University.

Mr. M. W. Pownall, University of Pennsylvania, has been appointed Instructor at Colgate University.

Professor G. B. Price, on leave from the University of Kansas, will be Visiting Professor at the California Institute of Technology for the academic year 1959–60.

Dr. P. B. Richards, Thompson Ramo Wooldridge, Inc., Cleveland, Ohio, has accepted a position as Physicist with the Missile and Space Vehicle Department of the General Electric Company, Philadelphia, Pennsylvania.

Mr. T. deF. Rogers, Jr., Emory University, has accepted a position as Associate Engineer with the Douglas Aircraft Company, Inc., Santa Monica, California.

Assistant Professor H. L. Rolf, Georgetown College, has been appointed Director of the Computer Center at Vanderbilt University.

Associate Professor Murray Rosenblatt, Indiana University, has been appointed Professor at Brown University.

Mr. W. B. Rundberg, San Jose State College, has been appointed a Teacher at Washington High School, Fremont, California.

Dr. P. T. Rygg, Iowa State University, has been appointed Assistant Professor at Montana State University.

Mr. N. S. Scarritt, Jr., University of Oklahoma, has been appointed Instructor at Purdue University.

Dr. J. R. Schue, Massachusetts Institute of Technology, has been appointed Assistant Professor at Oberlin College.

Dr. R. E. Seall was appointed Assistant Professor at Illinois Institute of Technology.

Associate Professor C. W. Seekins, U. S. Naval Academy, has been appointed Professor at Occidental College.

Associate Professor B. M. Seelbinder, University of Alabama, has been appointed Associate Professor at Wake Forest College.

Mrs. Beverly R. Sieg, University of Wisconsin, has been appointed Instructor at North Central College.

Assistant Professor R. W. Sloan, Carleton College, has been appointed Professor at the State University of New York, Teachers College at Oswego.

Rev. J. F. Smith, S.J., Catholic University of America, has accepted a position as Consultant at Woodstock College.

Dr. R. F. Smith has been appointed Professor at Earlham College.

Dr. J. J. Sopka, Jr., International Business Machines Corporation, New York, has accepted a position as Mathematician with the National Bureau of Standards, Boulder, Colorado.

Professor Emeritus R. C. Staley, University of North Dakota, has been appointed Visiting Professor at Macalester College.

Associate Professor E. M. Starr, Carnegie Institute of Technology, has retired.

Associate Professor Emeritus Ruth W. Stokes, Syracuse University, has been appointed Associate Professor at Longwood College for 1959–60.

Mr. R. K. Stump, Rutgers University, has been appointed Instructor at Muhlenberg College.

Mr. J. H. Taub, Rutgers University, has accepted a position as Associate Research Engineer with Boeing Airplane Company, Seattle, Washington.

Mr. R. L. Tennison, Oklahoma State University, has been appointed Assistant Professor at East Central State College.

Assistant Professor John Therrien, Lafayette College, has been appointed Assistant Professor at State University of New York, College for Teachers at Albany.

Professor W. J. Thomas, on leave from Baylor University, has been appointed Finance Examiner with the Texas Commission on Higher Education.

Dr. Leonard Tornheim, California Research Corporation, has been appointed Research Associate in Mathematics at the Richmond Laboratory.

Assistant Professor H. G. Tucker, on leave from the University of California, Riverside, has been appointed Assistant Research Statistician at the University of California, Berkeley.

Miss Carol S. Umbreit, University of Michigan, has been appointed a Junior High School Teacher with the Racine, Wisconsin Public Schools.

Assistant Professor H. S. Valk, on leave from the University of Oregon, has accepted a position as Physicist with the National Science Foundation.

Mr. E. R. Vance, Jr., State College of Washington, has accepted a position as Analyst with the General Electric Atomic Power Equipment Department, Pleasanton, California.

Mr. W. R. Volckhausen, University of Maryland, has been appointed Assistant Professor at Hampton Institute.

Professor O. E. Walder, South Dakota State College, has been promoted to Dean of Men.

Dr. G. L. Walker, American Optical Company, has been appointed Executive Director of the American Mathematical Society.

Mr. T. C. Walker, Browning, Montana, has accepted a position as Junior Engineer A, Boeing Airplane Company, Seattle, Washington.

Dr. L. E. Ward, Jr., Naval Ordnance Test Station, China Lake, California, has been appointed Associate Professor at the University of Oregon.

Dr. L. R. Welch, Jet Propulsion Laboratory, California Institute of Technology, has accepted a position as Mathematician with the Institute of Defense Analyses, Princeton, New Jersey.

Mr. A. L. Wilkinson, Texas A. & M. College, has accepted a position as Mathematician with the Department of Agriculture, Dallas, Texas.

Assistant Professor A. B. Willcox, Amherst College, has been promoted to Associate Professor.

Mr. E. R. Williams, Michigan State University, has accepted a position as Assistant Scientist "A" with Avco, Wilmington, Massachusetts.

Dr. J. C. Wilson, Case Institute of Technology, has been appointed Associate Professor at Central College, Pella, Iowa.

Mr. R. W. Wilson, Wisconsin State College, has been promoted to Assistant Professor.

Dr. P. B. Yale, Harvard University, has been appointed Assistant Professor at Oberlin College.

Assistant Professor L. N. Zaccaro, University of Rhode Island, has been appointed Associate Professor at Hiram College.

Dean Emeritus L. A. Howland, Wesleyan University, died on August 27, 1959. He was a member of the Association for thirty-eight years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

CONFERENCES FOR LECTURERS AT NSF 1959 SUMMER INSTITUTES IN MATHEMATICS

In March 1959, the National Science Foundation granted $58,000 to the MAA for the support of Conferences for Lecturers at 1959 Summer Institutes in Mathematics. President Allendoerfer had appointed a committee of the Association to organize these conferences, consisting of Professors E. G. Begle and G. B. Thomas, Chairman. Five regional conferences were held at the following places and times: Boston, April 11 and 12; Washington, D. C., April 18; Chicago, May 2 and 3; St. Louis, May 2; Palo Alto, May 2. A total of approximately 300 people attended these conferences.

The conferences were intended to improve the instruction in summer institutes through mutual discussions among instructors both experienced and inexperienced in institute operations, and to provide opportunity for exchange of information concerning the course content improvement activities of the Commission on Mathematics, the School Mathematics Study Group, the University of Illinois Mathematics Project, and the Association's Committee on the Undergraduate Program in Mathematics. At each of the five conferences were speakers representing each of these groups, a speaker with experience as an institute lecturer and a former participant in a summer institute.

It had been hoped that all lecturers at 1959 summer institutes in mathematics could be invited to one of the conferences. Because of the difficulty in securing this information in a brief time, invitations to the conferences were sent directly to the directors of summer institutes to transmit to the lecturers. This formula was used: in the case of an institute devoted solely to mathematics and intended for 50 or fewer participants, the director and one lecturer were invited. In the case of an institute directed solely to mathematics and involving more than 50 participants, the director and two lecturers were invited. For an institute involving mathematics in addition to other scientific subjects, one lecturer, in the field of mathematics, was invited. Since many of the directors had had experience in previous institutes, they were able to contribute much to the general discussion that took place at each conference.

Each of the organizations represented at the Conferences supplied printed or mimeographed material which could be used at a forthcoming summer institute. Each participant was given a summary of "Comments on NSF Summer Institutes," compiled by Professor Begle, which presented the reactions, experiences, and recommendations of lecturers at previous summer institutes.

A substantial number of the participants expressed appreciation to the Association and the National Science Foundation for arranging the conferences and commented that the conferences had been useful to them.

THE MAY MEETING OF THE INDIANA SECTION

The thirty-sixth annual spring meeting of the Indiana Section of the Mathematical Association of America was held Saturday, May 2, at Valparaiso University, Valparaiso, Indiana. Approximately 60 members attended. President G. N. Wollan of Purdue University, Chairman of the Section, presided at both the morning and afternoon sessions.

The following officers were elected: Chairman, Professor K. H. Carlson of Valparaiso University; Vice-Chairman, Professor M. E. Shanks of Purdue University; Secretary-Treasurer, Professor C. F. Brumfiel of Ball State Teachers College.

Professor Wollan reported upon the activity of the State School and College Committee. This committee is comprised of representatives of the Indiana Section of the Association and of the Indiana Council of Teachers of Mathematics. It represents a

cooperative endeavor on the part of high school and college teachers to study curriculum problems in mathematics in the elementary school, high school and college.

Professor Edwards, Chairman of the Committee on Awards, reported that one Association Medal had been awarded this year to a high school senior who exhibited high mathematical achievement in the Indiana Science Talent Search program.

The Annual High School Mathematics Contest, sponsored by the M.A.A. and the Society of Actuaries was discussed and it was agreed that the Indiana Section would continue to sponsor this test.

Professor Daniel Zelinsky of Northwestern University gave the invited hour address on "Tensor Products."

The following short papers were presented:

1. *Inverse functions vs. "converse" functions*, by Professor Joong Fang, Valparaiso University.

In general the mathematical inverse implies the identical in this sense, that if a function has an inverse, the latter is always able to undo whatever the former does. If "f" is considered an operator, "f^{-1}" is an inverse operator and $ff^{-1}(x) = f^{-1}f(x) = x$. The common practice in virtually all texts (*e.g.*, *Universal Mathematics* I, pp. 243–4) to produce the inverse function f^{-1} of a function f merely by interchanging the variables of f is thus entirely unwarranted. A new term "converse function" is recommended for such a case.

2. *A comment on the algebra of sets*, by Professor Joong Fang, Valparaiso University.

The *identity* set (in the proper sense) whose conspicuous absence has been either ignored or unsuspected, reveals its absurdity through the equivocality in inverse set-operations.

3. *Matrices over rings in which finitely generated ideals are principal—a survey*, by Professor Melvin Henriksen, Purdue University.

4. *Some remarks concerning the teaching of the Hilbert system*, by Professor Philip Dwinger, Purdue University.

The program of a course on "classical geometries" is outlined. After a critical discussion of the Euclidean system a rigorous treatment of the Hilbert system is given. The axiom of Pasch is presented in a stronger form. Instead of the axioms of congruence, the axioms of geometric displacements (Euclidean transformations) are introduced. Several models of non-Euclidean geometries are discussed. Particular attention is paid to hyperbolic geometry.

5. *Comments on a Notre Dame undergraduate mathematics program*, by Professor N. B. Haaser, University of Notre Dame, introduced by the Secretary.

The purpose of the program is to present elementary analysis both as mathematics and as an instrument of science and to do this in the spirit and the light of contemporary mathematics.

6. *Minimal fundamental sequences of functions*, by Professor Casper Goffman, Purdue University, introduced by the Secretary.

In a separable, metric, topological vector space, a sequence $\{f_n\}$ is fundamental if its finite linear combinations are dense in the space. It is minimal, if no proper subsequence is fundamental. Talalyan has shown that, for the space F of all measurable functions on $[0, 1]$, every fundamental sequence $\{f_n\}$ remains fundamental after any finite number of terms are deleted. This implies $\{f_n\}$ is universal in the sense that there are constants $\{a_n\}$ such that if $S_n = \sum_{k=1}^{n} a_k f_k$, then for every $f \in F$, a subsequence of $\{S_n\}$ converges a.e. to f. These results are shown here to be almost immediate consequences of the fact that the dual of F is trivial.

CHARLES BRUMFIEL, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The fifteenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Hartwick College, Oneonta, New York, on May 9, 1959. The Chairman of the Section, Professor Caroline A. Lester of the New

York State College for Teachers at Albany, presided at the morning session, and the Vice-Chairman, Professor Dis Maly of the Rensselaer Polytechnic Institute, presided at the afternoon session. There were 85 persons in attendance, including 53 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor Dis Maly, Rensselaer Polytechnic Institute; Vice-Chairman, Professor B. H. Gere, Hamilton College; Secretary-Treasurer, Professor N. G. Gunderson, University of Rochester. It was voted to continue the Committee on the Strengthening of Mathematics in the Section for another year. Professor Nura Turner of the New York State College for Teachers at Albany reported for the Contest Committee to the effect that the 1959 contest had been very successful. The Section voted to permit Professor Turner to use a small surplus which the Contest Committee had in a follow-up study of contest winners.

The program was as follows:

1. *Partial product calculators*, by Mr. E. I. Gale, Brockville Bible College.

A 10 by 90 matrix was displayed which could be used easily to determine the partial products in a multiplication. The matrix was then shown as a set of partial product "rods," which in a sense are generalizations of Napier's "bones" and Genaille's "rods." Next a model of a "mech-mult" was exhibited, this being an instrument in which the rods become continuous bands on a cylinder and partial products are read througn an adjustable slot.

2. *The honors course in mathematics at the University of Buffalo*, by Professor Edith R. Schneckenburger, University of Buffalo.

Several honors courses for freshmen have been introduced at the University of Buffalo during 1958–1959. Specifically mentioned was the honors course, Analytic Geometry and Calculus. The objectives of the course, unusual features such as the inclusion of calculus of functions of a complex variable, and conclusions regarding values of the program were presented.

3. *The NSF In-Service Institute at Oneonta State Teachers College*, by Professor C. E. Rusch, Oneonta State Teachers College.

A group of 30 junior and senior high school teachers of mathematics were members of an N.S.F. in-service institute at Oneonta during the school year 1958–1959. Two courses were offered, one in Statistics and one in Logic, Number, and Matrices. The courses were designed to contribute to the up-grading of mathematical instruction in the schools of this area.

4. *A report on the teaching of analytic geometry over closed circuit television*, by Professor Violet H. Larney, New York State Teachers College at Albany.

The author summarized the technique applied and results obtained when she taught analytic geometry to 200 students over closed circuit television in the fall semester, 1958. Students attended TV classes twice a week and small discussion sections, conducted by professors, once a week. Test results and a 25-question survey indicated that although TV students did as well academically as non-TV students, they listed many reasons why they were opposed to learning mathematics via television, even preferring large lecture hall classes. A statistical analysis revealed no significant difference between attitudes of the poor students and attitudes of the good students toward television classes.

5. *Changes in the certification climate*, by Dr. F. R. Kille, Associate Commissioner for Higher and Professional Education of the University of the State of New York. (By invitation.)

Commissioner Kille reviewed the history of the certification problem and outlined some of his views in an informal manner. He then introduced Mr. Alvin R. Lierheimer, Executive Secretary of the Certification Project, who outlined the work of the project and answered questions. A full report of the project will be available in the fall of 1959.

6. *Rings and modules*, by Professor R. E. Johnson, Smith College, M.A.A. Visiting Lecturer.

7. *Analytical Radon curves*, by Professor H. G. Helfenstein, University of Ottawa.

Radon has shown the existence of planar closed convex curves different from ellipses but having also "conjugate diameters" with the usual properties. The speaker showed that there exist such curves of arbitrarily high order of differentiability, but that only the ellipses remain if analyticity is required. There are applications of this result to Minkowskian and other metric spaces.

8. *The characterization of classes of orthogonal polynomials*, by Professor A. E. Danese, Union College.

Equivalent properties of characterization of the general class of orthogonal polynomials were discussed together with properties that characterize the classical orthogonal polynomials of Jacobi, Laguerre, and Hermite. These classical orthogonal polynomials were then generalized and equivalent characterization properties for these classes were discussed.

9. *A certain dicrete density function*, by Rev. M. A. Hanhauser, OFM, Siena College.

The speaker considered the problem of cards drawn from an ordinary deck without replacement until a spade appears. The discrete density function for the number of draws was proved to be $f(x)=13\binom{39}{x-1}/\binom{52}{x}$, $1\leq x\leq 40$. Induction was used to establsh that $\sum f(x)=1$.

10. *Polynomials and functions*, by Professor L. O. Kattsoff, Harpur College.

Starting with the ordinary definition of a polynomial over an integral domain, procedural rules were stated which make possible a transition from the notion of the "indeterminate" to that of a "variable." This allows the derivation of the concept of "function" from that of "polynomial" by means of mappings.

N. G. GUNDERSON, *Secretary*

CORRECTIONS

In *The Earle Raymond Hedrick Lectures*, this MONTHLY, vol. 66, 1959, p. 446, Professor L. H. Loomis is listed as from Howard University. This, of course, should be Harvard University.

In the report of the April meeting of the Iowa section, this MONTHLY, vol. 66, p. 628, the title of the paper by J. G. Baron, M.D., should be *On conditional probability*.

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

R. P. Agnew, A. N. Aheart, M. I. Aissen, H. L. Alder, C. B. Allendoerfer, B. H. Arnold, B. J. Attebery, S. P. Avann, J. P. Ballantine, J. H. Barrett, R. G. Bartle, R. A. Beaumont, E. G. Begle, R. E. Bellman, Garrett Birkhoff, David Blackwell, Richard Blum, L. M. Blumenthal, R. P. Boas, Samuel Borofsky, H. R. Brahana, Louis Brand, R. H. Bruck, S. J. Bryant, R. C. Buck, R. E. Burgess, L. Carlitz, J. W. Cell, C. C. Chang, D. G. Chapman, S. Chowla, Paul Civin, Nathan Coburn, Eckford Cohen, W. W. Comfort, J. A. Cooley, N. A. Court, H. S. M. Coxeter, Benjamin Dent, I. L. Doob, Frank Duttenhofer, H. G. Eggleston, Jaqueline Evans, D. J. Ewy, William Feller, H. E. Fettis, F. W. Ficken, N. J. Fine, Harley Flanders, L. Ford, G. E. Forsythe, J. S. Forsythe, Tomlinson Fort, A. L. Foster, Frank Gentry, Caspar Goffman, Michael Goldberg, Samuel Goldberg, D. B. Goodner, L. M. Graves, J. W. Green, Lewis Green, H. C. Griffith, W. C. Guenther, Marshall Hall, Jr., Paul Halmos, H. J. Hamilton, Frank Harary, S. M. Harmon, D. K. Harrison, Morton Hellman, Leon Henkin, Fritz Herzog, M. R. Hestenes, Edwin Hewitt, A. O. Hickson, F. B. Hildebrand, Franz Hohn, A. Horn, T. E. Hull, R. L. Iven, Nathan Jacobson, S. A. Jennings, R. E. Johnson, P. S. Jones, W. Kaplan, Houston Karnes, William Karush, John Kemeny, P. W. Ketchum,

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CALENDAR OF FUTURE MEETINGS

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 19–20, 1960.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

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Edited by HOWARD EVES, University of Maine and E. P. STARKE, Rutgers, The State University

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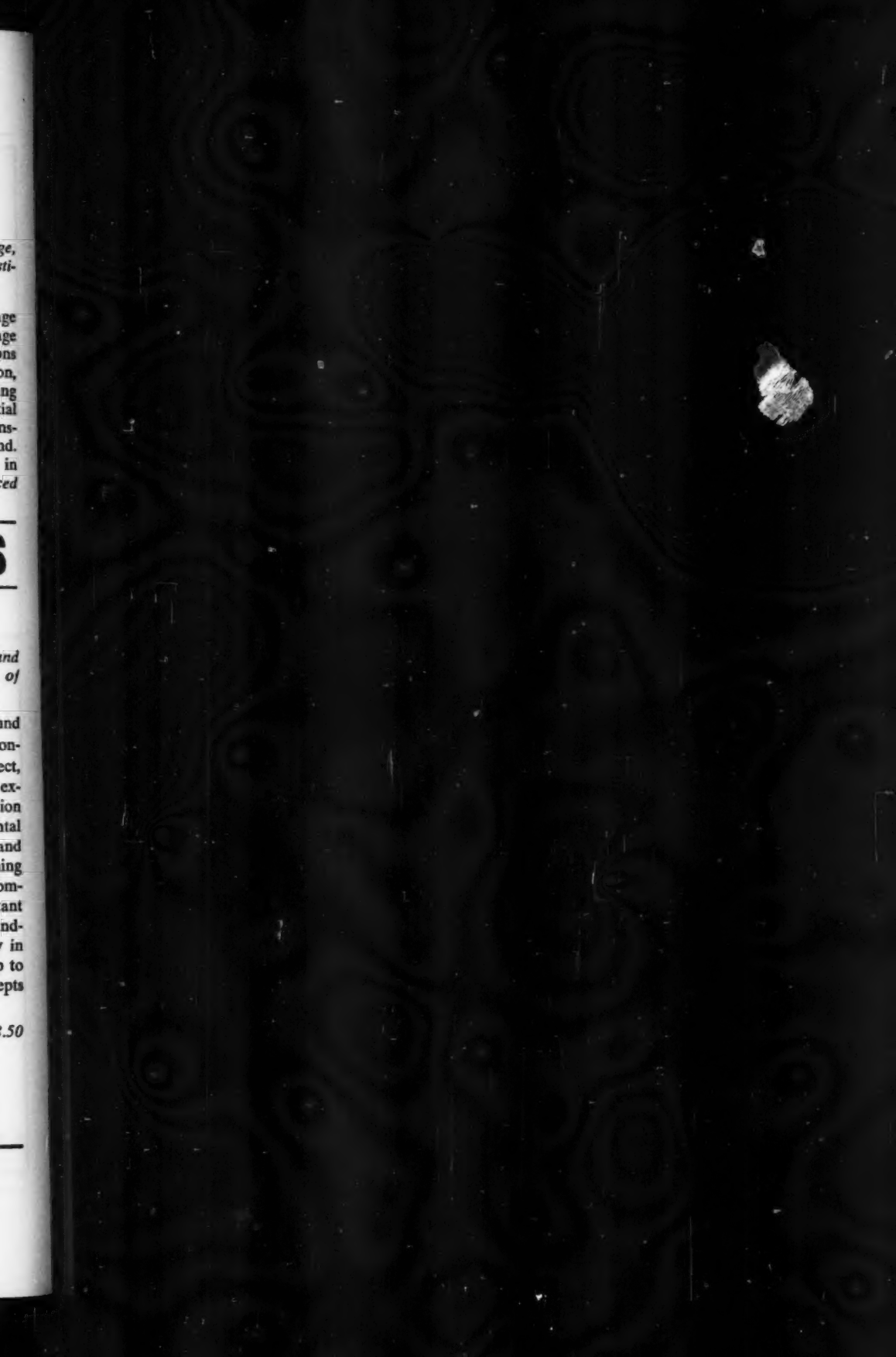

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